# On 3D and 1D Weyl particles in a 1D box 

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#### Abstract

We construct the most general families of self-adjoint boundary conditions for three (equivalent) Weyl Hamiltonian operators, each describing a threedimensional Weyl particle in a one-dimensional box situated along a Cartesian axis. These results are essentially obtained by using the most general family of selfadjoint boundary conditions for a Dirac Hamiltonian operator that describes a onedimensional Dirac particle in a box, in the Weyl representation, and by applying simple changes of representation to this operator. Likewise, we present the most general family of self-adjoint boundary conditions for a Weyl Hamiltonian operator that describes a one-dimensional Weyl particle in a one-dimensional box. We also obtain and discuss throughout the article distinct results related to the Weyl equations in $(3+1)$ and ( $1+1$ ) dimensions, in addition to their respective wave functions, and present certain key results related to representations for the Dirac equation in $(1+1)$ dimensions.


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## I. INTRODUCTION

In 1928, in the first edition of his book in German [1], and in 1929, in a couple of articles [2, 3], Weyl proposed - among other important things - two two-component wave equations for the description of free massless fermions in $(3+1)$ dimensions [4, 5] (an English translation of Ref. [3] can be seen in Ref. [6]). In 1957, Lee and Yang chose to assign one of these two equations specifically to the neutrino [7], but in 1958, Feynman and Gell-Mann showed that it was actually correct to assign the other equation to this particle [8]. Because there is now experimental evidence that a neutrino has a very small rest mass, the Weyl equations only approximately describe the behavior of this particle and its antiparticle. In passing, the (free) Weyl equations admit the standard minimal substitution and therefore admit an external electromagnetic four-potential; thus, these equations could also approximately describe the behavior of charged light fermions. In general, it can be said that the Weyl equations in $(3+1)$ dimensions describe three-dimensional Weyl particles (i.e., 3D Weyl particles), and in $(1+1)$ dimensions, they describe one-dimensional Weyl particles (i.e., 1D Weyl particles).

The Weyl equations in $(3+1)$ and $(1+1)$ dimensions are more easily constructed from the respective Dirac equation (in its respective Weyl representation). A particularly nice derivation of these equations in $(3+1)$ dimensions can be seen in Ref. [9], p. 79. In this reference, the Weyl equations were obtained by linearizing the van der Waerden second-order equation and then making the rest mass of the particle zero. Other derivations can be seen, for example, in Refs. [10-13].

Among other things, we want to explicitly obtain the most general family of self-adjoint boundary conditions for each of the three (equivalent) one-dimensional Cartesian reductions of the (three-dimensional) Weyl Hamiltonian operator (i.e., the usually named Dirac-Weyl Hamiltonian operator) for a 3D Weyl particle inside a three-dimensional square box. Each of these three operators describes a 3D Weyl particle that is ultimately restricted to a one-dimensional box of width $\ell$ situated along a Cartesian axis. Essentially, we obtain these three families of boundary conditions from the most general family of self-adjoint boundary conditions for a Dirac Hamiltonian operator that describes a 1D Dirac particle in a box of width $\ell$, in the Weyl representation. That is, we can obtain the families of boundary conditions for each of the three Weyl Hamiltonian operators by using only the aforementioned (Dirac) general family of boundary conditions and making some simple
changes of representation. All of these results are presented and developed in section III. Key results pertaining to representations for the Dirac equation in $(1+1)$ dimensions are also presented in this section. Before this, in section II, we present distinct results related to the Weyl equations in $(3+1)$ and $(1+1)$ dimensions and their respective wave functions. In section IV, we present the most general family of self-adjoint boundary conditions for a Weyl Hamiltonian operator that describes a 1D Weyl particle, also in a one-dimensional box of width $\ell$. Finally, our conclusions are presented in section V, and some results that complement what has been stated throughout the article are exhibited in the appendix.

## II. RESULTS PERTAINING TO THE 3D AND 1D WEYL EQUATIONS

## A. In (3+1) dimensions

The equation for a single free massless Dirac particle in $(3+1)$ dimensions has the form

$$
\begin{equation*}
\mathrm{i} \hat{\gamma}^{\mu} \partial_{\mu} \Psi=0 \tag{1}
\end{equation*}
$$

where $\Psi$ is the four-component Dirac wave function (or Dirac spinor), $\partial_{\mu}=\left(c^{-1} \partial_{t}, \nabla\right)$, and the Dirac gamma matrices $\hat{\gamma}^{\mu}$, with $\mu=0, j$, and $j=1,2,3$, satisfy the Clifford relation $\hat{\gamma}^{\mu} \hat{\gamma}^{\nu}+\hat{\gamma}^{\nu} \hat{\gamma}^{\mu}=2 g^{\mu \nu} \hat{1}_{4}$, where $g^{\mu \nu}=\operatorname{diag}(1,-1,-1,-1)$ is the metric tensor $\left(\hat{1}_{4}\right.$ is the $4 \times 4$ identity matrix). Additionally, we have the relation $\left(\hat{\gamma}^{\mu}\right)^{\dagger}=\hat{\gamma}^{0} \hat{\gamma}^{\mu} \hat{\gamma}^{0}$ ( $\dagger$ denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator, as usual). In the Weyl representation (or the chiral or spinor representation), the four-component wave function and the Dirac gamma matrices can be written as follows [14]:

$$
\Psi \equiv\left[\begin{array}{l}
\varphi_{1}  \tag{2}\\
\varphi_{2}
\end{array}\right], \quad \hat{\gamma}^{\mu}=\left[\begin{array}{cc}
\hat{0}_{2} & -\hat{\sigma}^{\mu} \\
-\hat{\sigma}^{\mu} & \hat{0}_{2}
\end{array}\right] .
$$

Likewise, the top two-component wave function can be written as $\varphi_{1} \equiv\left[\varphi_{1}^{\mathrm{t}} \varphi_{1}^{\mathrm{b}}\right]^{\mathrm{T}}$ and the bottom one as $\varphi_{2} \equiv\left[\varphi_{2}^{\mathrm{t}} \varphi_{2}^{\mathrm{b}}\right]^{\mathrm{T}}$, where $\varphi_{1}^{\mathrm{t}}$ and $\varphi_{2}^{\mathrm{t}}$ are the top components and $\varphi_{1}^{\mathrm{b}}$ and $\varphi_{2}^{\mathrm{b}}$ are the bottom components of the respective two-component wave function ( ${ }^{\mathrm{T}}$ represents the transpose of a matrix). In (3+1) dimensions, $\Psi$ is usually called the bispinor. Additionally, we have $\hat{\sigma}^{\mu} \equiv\left(\hat{1}_{2}, \hat{\sigma}_{x}, \hat{\sigma}_{y}, \hat{\sigma}_{z}\right) \equiv\left(\hat{1}_{2}, \hat{\boldsymbol{\sigma}}\right)$ and $\hat{\bar{\sigma}}^{\mu} \equiv\left(\hat{1}_{2},-\hat{\sigma}_{x},-\hat{\sigma}_{y},-\hat{\sigma}_{z}\right) \equiv\left(\hat{1}_{2},-\hat{\boldsymbol{\sigma}}\right)\left(\hat{1}_{2}\right.$ is the $2 \times 2$ identity matrix and $\hat{\sigma}_{j}$ 's are the usual Pauli matrices). Additionally, in Eq. (2), $\hat{0}_{2}$ is the 2-dimensional zero matrix.

By substituting $\Psi$ and $\hat{\gamma}^{\mu}$ from Eq. (2) into Eq. (1), we derive the well-known (explicitly covariant) two-component (free) Weyl equations, namely,

$$
\begin{equation*}
\mathrm{i} \hat{\sigma}^{\mu} \partial_{\mu} \varphi_{1}=0 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{i} \hat{\bar{\sigma}}^{\mu} \partial_{\mu} \varphi_{2}=0 \tag{4}
\end{equation*}
$$

where $\varphi_{1}$ and $\varphi_{2}$ are called Weyl spinors. Let us now forget how we obtained these two equations. As is known, Eq. (3) is usually assigned to the massless antineutrino and Eq. (4) to the massless neutrino [10]. However, it is possible that Eq. (4) only (or Eq. (3) only) is sufficient for the description of a massless fermion, a case called Weyl's two-component theory [9]. Moreover, Eqs. (3) and (4) are non-equivalent two-component equations in the sense that $\varphi_{1}$ and $\varphi_{2}$ transform in two different ways under a Lorentz boost, i.e., they transform according to two inequivalent representations of the Lorentz group [13] (although these two two-component wave functions transform in the same way under rotations). In fact, let us write the typical Lorentz boost with speed $v$ along the $x^{j}$-axis in the following way: $\left[c t^{\prime} x^{\prime} y^{\prime} z^{\prime}\right]^{\mathrm{T}}=\hat{\Lambda}_{j}[c t x y z]^{\mathrm{T}}$ (i.e., $x^{\mu \prime}=\left(\Lambda_{j}\right)_{\nu}^{\mu} x^{\nu} \Rightarrow x^{\mu}=\left(\Lambda_{j}^{-1}\right)^{\mu}{ }_{\nu} x^{\nu}{ }^{\prime}$ ), where

$$
\begin{equation*}
\hat{\Lambda}_{j}=\hat{\Lambda}_{j}(\omega)=\exp \left(\mathrm{i} \omega \hat{K}_{j}\right) \tag{5}
\end{equation*}
$$

with

$$
\hat{K}_{1}=\left[\begin{array}{cc}
\mathrm{i} \hat{\sigma}_{x} & \hat{0}_{2}  \tag{6}\\
\hat{0}_{2} & \hat{0}_{2}
\end{array}\right], \quad \hat{K}_{2}=\left[\begin{array}{cc}
\hat{0}_{2} & \frac{\mathrm{i}}{2}\left(\hat{1}_{2}+\hat{\sigma}_{z}\right) \\
\frac{\mathrm{i}}{2}\left(\hat{1}_{2}+\hat{\sigma}_{z}\right) & \hat{0}_{2}
\end{array}\right], \quad \hat{K}_{3}=\left[\begin{array}{cc}
\hat{0}_{2} & \frac{\mathrm{i}}{2}\left(\hat{\sigma}_{x}+\mathrm{i} \hat{\mathrm{\sigma}}_{y}\right) \\
\frac{\mathrm{i}}{2}\left(\hat{\sigma}_{x}-\mathrm{i} \hat{\sigma}_{y}\right) & \hat{0}_{2}
\end{array}\right],
$$

and, as usual, $\tanh (\omega)=v / c \equiv \beta$ and $\cosh (\omega)=\left(1-\beta^{2}\right)^{-1 / 2} \equiv \gamma$ (the speed of the primed reference frame in the direction of the $x^{j}$-axis with respect to the unprimed reference frame is precisely $v$ ). Then, under this Lorentz boost, the Dirac wave function in $(3+1)$ dimensions transforms as $\Psi^{\prime}\left(x^{j}, t^{\prime}\right)=\hat{S}\left(\Lambda_{j}\right) \Psi\left(x^{j}, t\right)$, where $\hat{S}\left(\Lambda_{j}\right)=\exp \left(-\omega \hat{\gamma}^{0} \hat{\gamma}^{j} / 2\right)$ and the latter matrix obeys the relation $\left(\Lambda_{j}\right)^{\mu}{ }_{\nu} \hat{\gamma}^{\nu}=\hat{S}^{-1}\left(\Lambda_{j}\right) \hat{\gamma}^{\mu} \hat{S}\left(\Lambda_{j}\right)$. Then, the $4 \times 4$ matrix $\hat{S}\left(\Lambda_{j}\right)$ is a block-diagonal matrix (as expected in the Weyl representation), and we obtain the following results [15]:

$$
\left[\begin{array}{l}
\varphi_{1}^{\mathrm{t}}\left(x^{j \prime}, t^{\prime}\right)  \tag{7}\\
\varphi_{1}^{\mathrm{b}}\left(x^{j{ }^{\prime}}, t^{\prime}\right)
\end{array}\right]=\left[\cosh \left(\frac{\omega}{2}\right) \hat{1}_{2}-\sinh \left(\frac{\omega}{2}\right) \hat{\sigma}_{j}\right]\left[\begin{array}{l}
\varphi_{1}^{\mathrm{t}}\left(x^{j}, t\right) \\
\varphi_{1}^{\mathrm{b}}\left(x^{j}, t\right)
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
\varphi_{2}^{\mathrm{t} \prime}\left(x^{j \prime}, t^{\prime}\right)  \tag{8}\\
\varphi_{2}^{\mathrm{b} \prime}\left(x^{j \prime}, t^{\prime}\right)
\end{array}\right]=\left[\cosh \left(\frac{\omega}{2}\right) \hat{1}_{2}+\sinh \left(\frac{\omega}{2}\right) \hat{\sigma}_{j}\right]\left[\begin{array}{l}
\varphi_{2}^{\mathrm{t}}\left(x^{j}, t\right) \\
\varphi_{2}^{\mathrm{b}}\left(x^{j}, t\right)
\end{array}\right] .
$$

Thus, we have two different kinds of two-component Weyl wave functions (or Weyl spinors) in $(3+1)$ dimensions. That is, the result in Eq. (7) is for a type of 3D Weyl particle, for example, for a massless antineutrino, and that in Eq. (8) is for the other type of 3D Weyl particle, for example, for a massless neutrino.

Usually, Eq. (3) is called the right-chiral Weyl equation and Eq. (4) is called the leftchiral Weyl equation because the four-component Dirac wave functions $\Psi_{+}=\left[\varphi_{1} 0\right]^{\mathrm{T}}=$ $\frac{1}{2}\left(\hat{1}_{4}+\hat{\gamma}^{5}\right) \Psi$ and $\Psi_{-}=\left[0 \varphi_{2}\right]^{\mathrm{T}}=\frac{1}{2}\left(\hat{1}_{4}-\hat{\gamma}^{5}\right) \Psi$ are eigenstates of the so-called chirality matrix $\hat{\gamma}^{5} \equiv \mathrm{i} \hat{\gamma}^{0} \hat{\gamma}^{1} \hat{\gamma}^{2} \hat{\gamma}^{3}=\operatorname{diag}\left(\hat{1}_{2},-\hat{1}_{2}\right)$, namely, $\hat{\gamma}^{5} \Psi_{ \pm}=( \pm 1) \Psi_{ \pm}\left(\Psi_{+}\right.$is the right-chiral eigenstate and $\Psi_{-}$is the left-chiral eigenstate). The chirality matrix is Hermitian and satisfies the relations $\left(\hat{\gamma}^{5}\right)^{2}=\hat{1}_{4}$, and $\hat{\gamma}^{5} \hat{\gamma}^{\mu}+\hat{\gamma}^{\mu} \hat{\gamma}^{5}=\operatorname{diag}\left(\hat{0}_{2}, \hat{0}_{2}\right)$ [15, 16]. Incidentally, because the matrices $\hat{S}\left(\Lambda_{j}\right)$ - boosts and rotations - commute with $\hat{\gamma}^{5}$, these Lorentz transformations do not change the chirality of a wave function.

Now, note that Eq. (1) can be written as $\left(\mathrm{i} \hbar \hat{\gamma}^{0} \partial_{0}-\hat{\gamma} \cdot \hat{\mathbf{p}}\right) \Psi=0$, where $\hat{\mathbf{p}}=-\mathrm{i} \hbar \hat{1}_{4} \nabla$ is the (Hermitian) Dirac momentum operator in (3+1) dimensions, and $\hat{\gamma}=\left(\hat{\gamma}^{1}, \hat{\gamma}^{2}, \hat{\gamma}^{3}\right)$. The latter equation can also be written as $\left(\mathrm{i} \hbar \hat{\gamma}^{5} \partial_{0}-\hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}\right) \Psi=0$, where $\hat{\boldsymbol{\Sigma}}=\hat{\gamma}^{5} \hat{\gamma}^{0} \hat{\boldsymbol{\gamma}}=\hat{\boldsymbol{\Sigma}}^{\dagger}$. Now, assuming that the operator $\mathrm{i} \hbar \hat{\gamma}^{5} \partial_{0}-\hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}}$ is acting on a typical plane-wave solution $\Psi_{\mathbf{p}}$, we obtain the algebraic relation $\left(\operatorname{sgn}(E)\|\mathbf{p}\| \hat{\gamma}^{5}-\hat{\boldsymbol{\Sigma}} \cdot \mathbf{p}\right) \Psi_{\mathbf{p}}=0$, where $\operatorname{sgn}(E)$ is the sign of the relativistic energy of the massless Dirac particle, i.e., $E=\operatorname{sgn}(E) c\|\mathbf{p}\|$. Thus, the (Hermitian) matrix $\hat{\boldsymbol{\Sigma}} \cdot \mathbf{p} /\|\mathbf{p}\|$ is related to $\hat{\gamma}^{5}$ via the aforementioned algebraic relation. More generally, we could say that the (Hermitian) operator $\hat{\lambda} \equiv \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}} /\|\mathbf{p}\|=\operatorname{diag}\left(\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|, \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|\right)$ and the chirality matrix are linked (in the latter relation, $\mathbf{p}_{[2]}$ indicates the eigenvalues of the operator $\hat{\mathbf{p}}_{[2]} \equiv-\mathrm{i} \hbar \hat{1}_{2} \nabla$, namely, the Weyl momentum operator, and $\left.\left\|\mathbf{p}_{[2]}\right\|=\|\mathbf{p}\|\right)$. Finally, as is known, the matrix $\hat{\Sigma}^{j} / 2$ is the infinitesimal generator of rotation through an angle $\theta$ around the $x^{j}$-axis, and this matrix is related to the generalization of the spin operator in $(3+1)$ dimensions, $\hat{S}^{j}=\hbar \hat{\Sigma}^{j} / 2$ (see appendix, subsection A). Thus, the operator $\hat{\lambda}$ pertains to the projection of the spin onto the direction of momentum (which is not necessarily the direction of the particle motion) and is called the helicity operator (see appendix, subsection B).

Observe that, because $\operatorname{sgn}(E) \hat{\gamma}^{5} \Psi_{\mathbf{p}}=\hat{\lambda} \Psi_{\mathbf{p}}, \Psi_{\mathbf{p}}=\left(\Psi_{+}\right)_{\mathbf{p}}+\left(\Psi_{-}\right)_{\mathbf{p}}$ and $\hat{\gamma}^{5}\left(\Psi_{ \pm}\right)_{\mathbf{p}}=$
$( \pm 1)\left(\Psi_{ \pm}\right)_{\mathbf{p}}$, we have $\hat{\lambda}\left(\Psi_{+}\right)_{\mathbf{p}}=\operatorname{sgn}(E) \hat{1}_{4}\left(\Psi_{+}\right)_{\mathbf{p}}$ and $\hat{\lambda}\left(\Psi_{-}\right)_{\mathbf{p}}=-\operatorname{sgn}(E) \hat{1}_{4}\left(\Psi_{-}\right)_{\mathbf{p}}$. The eigenstate of $\hat{\lambda}$ with eigenvalue +1 is called the right-handed (or right-helical) state (spin parallel to momentum), and the eigenstate of $\hat{\lambda}$ with eigenvalue -1 is called the left-handed (or lefthelical) state (spin opposite to momentum) [16]. Clearly, for positive energies, right-handed and right-chiral, as well as left-handed and left-chiral, can be considered as similar concepts (naturally, within the present discussion), i.e., $\left(\Psi_{+}\right)_{\mathbf{p}}$ and $\left(\Psi_{-}\right)_{\mathbf{p}}$ are eigenstates of $\hat{\lambda}$ and $\hat{\gamma}^{5}$ with eigenvalues +1 and -1 , respectively. From the two eigenvalue equations for the operator $\hat{\lambda}$, two eigenvalue equations for the operator $\hat{\lambda}_{[2]} \equiv \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|$ are obtained, namely, $\hat{\lambda}_{[2]}\left(\varphi_{1}\right)_{\mathbf{p}}=\operatorname{sgn}(E) \hat{1}_{2}\left(\varphi_{1}\right)_{\mathbf{p}}$ and $\hat{\lambda}_{[2]}\left(\varphi_{2}\right)_{\mathbf{p}}=-\operatorname{sgn}(E) \hat{1}_{2}\left(\varphi_{2}\right)_{\mathbf{p}}$. As expected, the latter two equations are precisely Eqs. (3) and (4) (with the latter for plane-wave eigensolutions). Because $\hbar \hat{\boldsymbol{\sigma}} / 2\left(\equiv \hat{\mathbf{S}}_{[2]}\right)$ is the spin operator for two-component wave functions, the operator $\hat{\lambda}_{[2]}$ can be considered as the helicity operator for this type of wave function. Thus, assuming or postulating that $\left(\varphi_{1}\right)_{\mathbf{p}}$ describes the antineutrino, we can say that the helicity of this positive-energy particle is positive, i.e., the antineutrino is right-handed (the latter fact has been determined experimentally). However, the same equation for $\left(\varphi_{1}\right)_{\mathbf{p}}$ also tells us that the helicity of the negative-energy antineutrino is negative, and according to the so-called hole theory, this last result is interpreted as the helicity of the positive-energy neutrino also being negative; thus, the neutrino would be left-handed (the latter fact has also been determined experimentally) [9, 10]. Alternatively, assuming that $\left(\varphi_{2}\right)_{\mathbf{p}}$ describes the neutrino, we can say that the helicity of this positive-energy particle is negative; thus, the conclusion is yet again that the neutrino is left-handed. The same equation for $\left(\varphi_{2}\right)_{\mathbf{p}}$ also tells us that the helicity of the negative-energy neutrino is positive, and according to hole theory, this last result is interpreted as the helicity of the positive-energy antineutrino also being positive; thus, the conclusion is yet again that the antineutrino is right-handed [9, 10].

Finally, as is known, the Dirac equation (1) can provide real-valued solutions as long as the Dirac gamma matrices satisfy $\left(\mathrm{i}^{\mu}\right)^{*}=\mathrm{i} \hat{\gamma}^{\mu}$; this is precisely the condition that defines the Majorana representation of the Dirac matrices (the asterisk * denotes a complex conjugate, as usual). On the other hand, because the term $\mathrm{i} \hat{\sigma}^{0}$ in the Weyl equation (3) satisfies (i $\left.\hat{\sigma}^{0}\right)^{*}=$ $-\mathrm{i} \hat{\sigma}^{0}$, this equation can give real-valued solutions only if the matrices $\hat{\sigma}^{\mu}$ for $\mu=1,2,3$ in Eq. (3) also satisfy $\left(\mathrm{i} \hat{\sigma}^{\mu}\right)^{*}=-\mathrm{i} \hat{\sigma}^{\mu}$, in which case we obtain $\mu=1,3$ (the latter results for the matrices $\hat{\sigma}^{\mu}$ are obviously also valid for the matrices $\hat{\bar{\sigma}}^{\mu}$ present in the Weyl equation (4)). Thus, although the (free) Weyl equations are considered to describe massless neutral
fermions, the solutions of these equations are not always real. For example, the solutions $\varphi_{1}=\varphi_{1}(y, t)\left(\Rightarrow \partial_{1} \varphi_{1}=\partial_{3} \varphi_{1}=0\right)$ of Eq. (3), i.e., of the equation $\left(\mathrm{i} \hat{\sigma}^{0} \partial_{0}+\mathrm{i} \hat{\sigma}^{2} \partial_{2}\right) \varphi_{1}=0$, are always complex-valued. The same goes for the solutions $\varphi_{2}=\varphi_{2}(y, t)$ of Eq. (4) (in this case, Eq. (4) is $\left(\mathrm{i} \hat{\bar{\sigma}}^{0} \partial_{0}+\mathrm{i} \hat{\bar{\sigma}}^{2} \partial_{2}\right) \varphi_{2}=0$ ). Certainly, because the Weyl equations (Eqs. (3) and (4)) can explicitly arise by using the Weyl representation in the Dirac equation (Eq. (1)), it is expected that these equations will also generate complex solutions. If the Majorana representation is used in the Dirac equation, then a real system of coupled equations is obtained (which is why it admits real-valued solutions), not two non-equivalent explicitly covariant complex first-order equations [14].

## B. In $(1+1)$ dimensions

The equation for a single free massless Dirac particle in (1+1) dimensions (or the onedimensional free massless Dirac particle) has the form

$$
\begin{equation*}
\mathrm{i} \hat{\gamma}^{\mu} \partial_{\mu} \Psi=0, \tag{9}
\end{equation*}
$$

where $\Psi$ is a two-component wave function, and the Dirac matrices $\hat{\gamma}^{\mu}$, with $\mu=0,1$, satisfy the relations $\hat{\gamma}^{\mu} \hat{\gamma}^{\nu}+\hat{\gamma}^{\nu} \hat{\gamma}^{\mu}=2 g^{\mu \nu} \hat{1}_{2}$, where $g^{\mu \nu}=\operatorname{diag}(1,-1)\left(\hat{1}_{2}\right.$ is the $2 \times 2$ identity matrix), and $\left(\hat{\gamma}^{\mu}\right)^{\dagger}=\hat{\gamma}^{0} \hat{\gamma}^{\mu} \hat{\gamma}^{0}$. Here, we utilize the coordinates $x^{0}=c t$ and $x^{1}=x$ and $\partial_{\mu}=\left(c^{-1} \partial_{t}, \partial_{x}\right)$, as usual. In the Weyl representation, the two-component wave function and the Dirac matrices can be written as follows [14]:

$$
\Psi \equiv\left[\begin{array}{l}
\varphi_{1}  \tag{10}\\
\varphi_{2}
\end{array}\right], \quad \hat{\gamma}^{0}=\hat{\sigma}_{x}, \quad \hat{\gamma}^{1}=-\mathrm{i} \hat{\sigma}_{y} .
$$

By substituting $\Psi, \hat{\gamma}^{0}$ and $\hat{\gamma}^{1}$ from Eq. (10) into Eq. (9), we derive two one-component equations, namely,

$$
\begin{equation*}
\mathrm{i}\left(\partial_{0}+\partial_{1}\right) \varphi_{1}=0, \quad \text { and } \quad \mathrm{i}\left(\partial_{0}-\partial_{1}\right) \varphi_{2}=0 \tag{11}
\end{equation*}
$$

These two equations would be the (free) Weyl equations in (1+1) dimensions [17], and each of them would describe - let us say - massless neutral one-dimensional "fermions" (i.e., 1D uncharged Weyl particles). Additionally, the one-component wave functions $\varphi_{1}$ and $\varphi_{2}$ are transformed in two different ways under the Lorentz boost. In effect, let us write this Lorentz boost with speed $v$ along the $x$-axis in the following way: $\left[c t^{\prime} x^{\prime}\right]^{\mathrm{T}}=\hat{\Lambda}[c t x]^{\mathrm{T}}$ (i.e.,
$\left.x^{\mu \prime}=\Lambda^{\mu}{ }_{\nu} x^{\nu} \Rightarrow x^{\mu}=\left(\Lambda_{j}^{-1}\right)^{\mu}{ }_{\nu} x^{\nu \prime}\right)$, where $\hat{\Lambda}=\hat{\Lambda}(\omega)=\exp (\mathrm{i} \omega \hat{K})$, with $\hat{K}=\mathrm{i} \hat{\sigma}_{x}, \tanh (\omega)=$ $v / c \equiv \beta$ and $\cosh (\omega)=\left(1-\beta^{2}\right)^{-1 / 2} \equiv \gamma(\omega \in \mathbb{R})$. Then, under this transformation, the Dirac wave function in $(1+1)$ dimensions transforms as $\Psi^{\prime}\left(x^{\prime}, t^{\prime}\right)=\hat{S}(\Lambda) \Psi(x, t)$, where $\hat{S}(\Lambda)=\exp \left(-\omega \hat{\gamma}^{0} \hat{\gamma}^{1} / 2\right)=\exp \left(-\omega \hat{\sigma}_{z} / 2\right)$, and the latter matrix obeys the relation $\Lambda^{\mu}{ }_{\nu} \hat{\gamma}^{\nu}=$ $\hat{S}^{-1}(\Lambda) \hat{\gamma}^{\mu} \hat{S}(\Lambda)$. Then, the matrix $\hat{S}(\Lambda)$ is a diagonal matrix (as expected), and we obtain the following results:

$$
\begin{equation*}
\varphi_{1}^{\prime}\left(x^{\prime}, t^{\prime}\right)=\left[\cosh \left(\frac{\omega}{2}\right)-\sinh \left(\frac{\omega}{2}\right)\right] \varphi_{1}(x, t) \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi_{2}^{\prime}\left(x^{\prime}, t^{\prime}\right)=\left[\cosh \left(\frac{\omega}{2}\right)+\sinh \left(\frac{\omega}{2}\right)\right] \varphi_{2}(x, t) . \tag{13}
\end{equation*}
$$

Thus, we have two different types of one-component Weyl wave function in (1+1) dimensions. That is, the result in Eq. (12) is for one type of 1D Weyl particle, and that of Eq. (13) is for the other type of 1D Weyl particle. On the other hand, by comparing the relations in (7) and (8) with those in (12) and (13), we see that the two-component wave function for a one-dimensional Dirac particle could transform in a similar way to the two-component wave function for a specific type of 3D Weyl particle. This is the case, for example, when the one-dimensional Dirac particle is constrained to the $z$-axis, in which case its respective wave function, $\Psi(z, t) \equiv\left[\varphi_{1}(z, t) \varphi_{2}(z, t)\right]^{\mathrm{T}}$, is exactly transformed as the Weyl wave function $\varphi_{1}(z, t) \equiv\left[\varphi_{1}^{\mathrm{t}}(z, t) \varphi_{1}^{\mathrm{b}}(z, t)\right]^{\mathrm{T}}$ (see Eq. (7)). The Dirac wave function $\Psi(z, t)$ would also transform as the Weyl wave function $\varphi_{2}(z, t) \equiv\left[\varphi_{2}^{\mathrm{t}}(z, t) \varphi_{2}^{\mathrm{b}}(z, t)\right]^{\mathrm{T}}$ if the replacement $\omega \rightarrow$ $-\omega$ is made in the function given by Eq. (8) (i.e., if the relative speed between the Lorentz frames changes from $v$ to $-v$ ).

In $(1+1)$ dimensions, the matrix $\hat{\Gamma}^{5} \equiv-\mathrm{i} \hat{\gamma}_{[2]}^{5}$, where $\hat{\gamma}_{[2]}^{5} \equiv \mathrm{i} \hat{\gamma}^{0} \hat{\gamma}^{1}$, is the chirality matrix, i.e., $\hat{\Gamma}^{5}=\hat{\sigma}_{z}$. As we know, this matrix is Hermitian and satisfies the relations $\left(\hat{\Gamma}^{5}\right)^{2}=\hat{1}_{2}$ and $\hat{\Gamma}^{5} \hat{\gamma}^{\mu}+\hat{\gamma}^{\mu} \hat{\Gamma}^{5}=\hat{0}_{2}$ [14, 18]. Thus, the $(2 \times 2)$ chirality matrix in $(1+1)$ dimensions satisfies the same (three) basic properties as the $(4 \times 4)$ chirality matrix in $(3+1)$ dimensions, as should be the case. Evidently, the two-component Dirac wave functions $\Psi_{+}=\left[\varphi_{1} 0\right]^{\mathrm{T}}=\frac{1}{2}\left(\hat{1}_{2}+\hat{\Gamma}^{5}\right) \Psi$ (which must also satisfy the relations $\frac{1}{2}\left(\hat{1}_{2}+\hat{\Gamma}^{5}\right) \Psi_{+}=\Psi_{+}$and $\frac{1}{2}\left(\hat{1}_{2}-\hat{\Gamma}^{5}\right) \Psi_{+}=0$ ) and $\Psi_{-}=\left[0 \varphi_{2}\right]^{\mathrm{T}}=\frac{1}{2}\left(\hat{1}_{2}-\hat{\Gamma}^{5}\right) \Psi$ (which must also satisfy the relations $\frac{1}{2}\left(\hat{1}_{2}-\hat{\Gamma}^{5}\right) \Psi_{-}=\Psi_{-}$ and $\left.\frac{1}{2}\left(\hat{1}_{2}+\hat{\Gamma}^{5}\right) \Psi_{-}=0\right)$ are eigenstates of $\hat{\Gamma}^{5}$. Again, $\Psi_{+}$is called the right-chiral eigenstate (eigenvalue +1 ) and $\Psi_{-}$the left-chiral eigenstate (eigenvalue -1 ). Certainly, because the
matrices $\hat{S}(\Lambda)$ and $\hat{\Gamma}^{5}$ commute, the Lorentz boost does not change the chirality of the wave function.

Note that Eq. (9) can be written as $\left(\mathrm{i} \hbar \hat{\gamma}^{0} \partial_{0}-\hat{\gamma}^{1} \hat{\mathrm{p}}\right) \Psi=0$, where $\hat{\mathrm{p}}=-\mathrm{i} \hbar \hat{1}_{2} \partial_{1}$ is the (Hermitian) Dirac momentum operator in $(1+1)$ dimensions. The latter equation can also be written as $\left(\mathrm{i} \hbar \hat{\Gamma}^{5} \partial_{0}-\hat{\mathrm{p}}\right) \Psi=0$ (remember that $\hat{\Gamma}^{5}=\hat{\gamma}^{0} \hat{\gamma}^{1}$ ). Now, assuming that the operator $\mathrm{i} \hbar \hat{\Gamma}^{5} \partial_{0}-\hat{\mathrm{p}}$ is acting on a common plane-wave eigensolution $\Psi_{\mathrm{p}}$, we obtain the algebraic relation $\left(\operatorname{sgn}(E)|\mathrm{p}| \hat{\Gamma}^{5}-\mathrm{p} \hat{1}_{2}\right) \Psi_{\mathrm{p}}=0$, where $\operatorname{sgn}(E)$ is the sign of the relativistic energy of the one-dimensional massless Dirac particle, i.e., $E=\operatorname{sgn}(E) c|\mathrm{p}|$. Thus, in $(1+1)$ dimensions, the $2 \times 2$ matrix $p \hat{1}_{2} /|\mathrm{p}|$ is related to $\hat{\Gamma}^{5}$ via the aforementioned algebraic relation. We could say far more generally that there is a connection between the operator $\hat{\mathrm{p}} /|\mathrm{p}|=\operatorname{diag}\left(\hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|, \hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|\right)$ and the chirality matrix (in the latter expression, $\mathrm{p}_{[1]}$ indicates the eigenvalues of the operator $\hat{\mathrm{p}}_{[1]} \equiv-\mathrm{i} \hbar \partial_{1}$, namely, the momentum operator that acts on one-component wave functions, and $\left.\left|\mathrm{p}_{[1]}\right|=|\mathrm{p}|\right)$.

Note that, because $\operatorname{sgn}(E) \hat{\Gamma}^{5} \Psi_{\mathrm{p}}=(\hat{\mathrm{p}} /|\mathrm{p}|) \Psi_{\mathrm{p}}, \Psi_{\mathrm{p}}=\left(\Psi_{+}\right)_{\mathrm{p}}+\left(\Psi_{-}\right)_{\mathrm{p}}$ and $\hat{\Gamma}^{5}\left(\Psi_{ \pm}\right)_{\mathrm{p}}=$ $( \pm 1)\left(\Psi_{ \pm}\right)_{\mathrm{p}}$, we have that $(\hat{\mathrm{p}} /|\mathrm{p}|)\left(\Psi_{+}\right)_{\mathrm{p}}=\operatorname{sgn}(E) \hat{1}_{2}\left(\Psi_{+}\right)_{\mathrm{p}} \quad$ and $(\hat{\mathrm{p}} /|\mathrm{p}|)\left(\Psi_{-}\right)_{\mathrm{p}}=$ $-\operatorname{sgn}(E) \hat{1}_{2}\left(\Psi_{-}\right)_{\mathrm{p}}$. Thus, in $(1+1)$ dimensions, for positive energies, $\left(\Psi_{+}\right)_{\mathrm{p}}$ and $\left(\Psi_{-}\right)_{\mathrm{p}}$ are eigenstates of $\hat{\Gamma}^{5}$ and $\hat{\mathrm{p}} /|\mathrm{p}|$ with eigenvalues +1 and -1 , respectively. From the latter two eigenvalue equations, two other eigenvalue equations for the operator $\hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|$ are obtained, namely, $\left(\hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|\right)\left(\varphi_{1}\right)_{\mathrm{p}}=\operatorname{sgn}(E)\left(\varphi_{1}\right)_{\mathrm{p}}$ and $\left(\hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|\right)\left(\varphi_{2}\right)_{\mathrm{p}}=-\operatorname{sgn}(E)\left(\varphi_{2}\right)_{\mathrm{p}}$. Certainly, the latter two equations are the equations in (11) for the eigenstates of the energy and momentum. Thus, in $(1+1)$ dimensions, the momentum operator $\hat{\mathrm{p}}_{[1]}$ plays a role somewhat similar to that of the helicity operator $\hat{\lambda}_{[2]}$ in $(3+1)$ dimensions (see appendix, subsection B). Naturally, in $(1+1)$ dimensions, there is no such thing as helicity or spin. Note that the momentum of the positive-energy 1D Weyl particle described with the wave function $\left(\varphi_{1}\right)_{\mathrm{p}}$ is positive, and if it has negative energy, then its momentum is negative. Likewise, the momentum of the other positive-energy 1D Weyl particle, described with the wave function $\left(\varphi_{2}\right)_{\mathrm{p}}$, is negative, and if it has negative energy, then its momentum is positive. In (1+1) dimensions, the (Dirac) chiral plane-wave eigenstates $\left(\Psi_{+}\right)_{p}$ and $\left(\Psi_{-}\right)_{p}$ are such that the charge conjugate of $\left(\Psi_{+}\right)_{\mathrm{p}}\left(\left(\Psi_{-}\right)_{\mathrm{p}}\right)$ is also a right-chiral (left-chiral) state [14]. Thus, the one-component states $\left(\varphi_{1}\right)_{\mathrm{p}}$ and $\left(\varphi_{2}\right)_{\mathrm{p}}$ would not exactly describe a particle-antiparticle pair.

It should be noted, in passing, that unlike what happens with the Weyl equations in $(3+1)$ dimensions (Eqs. (3) and (4)), the Weyl equations in $(1+1)$ dimensions (equations
in (11)) can always give real solutions, i.e., the latter equations can have solutions à la Majorana. Certainly, complex solutions can also be obtained.

Now, let us return to the Dirac equation in (9). Certainly, in choosing a representation, one is choosing a set of Dirac matrices that satisfies the Clifford relation. In addition to $(i)$, the Weyl (or chiral or spinor) representation, $\left\{\hat{\gamma}^{0}, \hat{\gamma}^{1}\right\}=\left\{\hat{\sigma}_{x},-\mathrm{i} \hat{\sigma}_{y}\right\}$ (with the Dirac wave function in Eq. (9) written as $\Psi \equiv\left[\varphi_{1} \varphi_{2}\right]^{\mathrm{T}}$ ), we must consider three other representations in ( $1+1$ ) dimensions, namely, (ii) the Dirac (or standard or Dirac-Pauli) representation, $\left\{\hat{\gamma}^{0}, \hat{\gamma}^{1}\right\}=\left\{\hat{\sigma}_{z}, \mathrm{i} \hat{\sigma}_{y}\right\}$ (in this case, we write $\Psi \equiv[\varphi \chi]^{\mathrm{T}}$ ); (iii) the Majorana representation, $\left\{\hat{\gamma}^{0}, \hat{\gamma}^{1}\right\}=\left\{\hat{\sigma}_{y},-\mathrm{i} \hat{\sigma}_{z}\right\}$ (in this case, $\Psi \equiv\left[\phi_{1} \phi_{2}\right]^{\mathrm{T}}$ ); and (iv) the Jackiw-Rebbi representation, $\left\{\hat{\gamma}^{0}, \hat{\gamma}^{1}\right\}=\left\{\hat{\sigma}_{x}, \mathrm{i} \hat{\sigma}_{z}\right\}$ (in this case, $\Psi \equiv\left[\chi_{1} \chi_{2}\right]^{\mathrm{T}}$ ) [19-21]. In the next section, we present some connections between these representations, i.e., the connections between the Dirac matrices and the two-component wave functions in two different representations.

## III. BOUNDARY CONDITIONS FOR THE 3D WEYL PARTICLE IN A 1D BOX

The Weyl equations (3) and (4), written compactly in Hamiltonian form, are

$$
\begin{equation*}
\mathrm{i} \hbar \hat{1}_{2} \frac{\partial}{\partial t} \varphi_{a}=\hat{\mathrm{H}}_{a} \varphi_{a}, \quad a=1,2 \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\mathrm{H}}_{a} \equiv \sum_{j=1}^{j=3} \hat{\mathrm{H}}_{a, j}=-\mathrm{i} \hbar c(-1)^{a-1} \sum_{j=1}^{j=3} \hat{\sigma}_{j} \frac{\partial}{\partial x^{j}}=-\mathrm{i} \hbar c(-1)^{a-1} \hat{\boldsymbol{\sigma}} \cdot \nabla \tag{15}
\end{equation*}
$$

is the formally self-adjoint, or Hermitian, Weyl Hamiltonian operator (or Dirac-Weyl operator), i.e., $\hat{\mathrm{H}}_{a}=\hat{\mathrm{H}}_{a}^{\dagger}$ ( $\dagger$ denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator). Naturally, this operator can also be written in terms of the spin operator $\hat{\mathbf{S}}_{[2]}$, namely, $\hat{\mathrm{H}}_{a}=c \hat{\mathbf{S}}_{[2]} \cdot \hat{\mathbf{p}}_{[2]} /(-1)^{a-1} \frac{\hbar}{2}$ (as we know, the label $a$ indicates a particular type of 3D Weyl particle). The latter expression for the operator $\hat{H}_{a}$ could be generalized to the case of arbitrary spin (see Ref. [4] and references therein). In general, $\hat{\mathrm{H}}_{a}$ acts on the twocomponent Weyl wave functions $\varphi_{a}=\varphi_{a}(\mathbf{r}, t)$ that belong to the Hilbert space of square integrable functions, $\mathcal{H}=L^{2}(\Omega)^{2}$, where $\Omega \subset \mathbb{R}^{3}$ represents a volume in three-dimensional space. The scalar product in $\mathcal{H}$ is denoted by $\left\langle\psi_{a}, \chi_{a}\right\rangle \equiv \int_{\Omega} \mathrm{d}^{3} \mathbf{r} \psi_{a}^{\dagger} \chi_{a}$, and the norm is $\left\|\psi_{a}\right\| \equiv \sqrt{\left\langle\psi_{a}, \psi_{a}\right\rangle}$, as usual. The domain of the Hamiltonian, $\mathcal{D}\left(\hat{\mathrm{H}}_{a}\right)$, is the set of Weyl wave functions in $\mathcal{H}$ on which $\hat{\mathrm{H}}_{a}$ can act and (generally) includes the boundary conditions that these wave functions must satisfy at the boundary of the volume $\Omega$. Additionally, the

Hamiltonian, when acting on one of these functions, must produce a function that belongs to $\mathcal{H}$. By virtue of one integration by parts, the self-adjointness condition (and therefore the hermiticity condition) of $\hat{\mathrm{H}}_{a}$ is obtained, namely,

$$
\begin{equation*}
\left\langle\psi_{a}, \hat{\mathrm{H}}_{a} \chi_{a}\right\rangle=\left\langle\hat{\mathrm{H}}_{a} \psi_{a}, \chi_{a}\right\rangle-\mathrm{i} \hbar c(-1)^{a-1} \int_{\Omega} \mathrm{d}^{3} \mathbf{r} \nabla \cdot\left(\psi_{a}^{\dagger} \hat{\boldsymbol{\sigma}} \chi_{a}\right)=\left\langle\hat{\mathrm{H}}_{a} \psi_{a}, \chi_{a}\right\rangle \tag{16}
\end{equation*}
$$

where the volume integral is the surface integral over the boundary of the volume, i.e., $\oint_{\partial \Omega} \psi^{\dagger} \hat{\boldsymbol{\sigma}} \chi \cdot \mathrm{d} \boldsymbol{S}$ (because of the Gauss-Ostrogradsky theorem), and it must vanish because one imposes some specific boundary conditions on $\psi_{a}$ and $\chi_{a}$ at $\partial \Omega$ that belong to $\mathcal{D}\left(\hat{\mathrm{H}}_{a}\right)=$ $\mathcal{D}\left(\hat{\mathrm{H}}_{a}^{\dagger}\right)$. Remember that, given $\hat{\mathrm{H}}_{a}$, the relation $\left\langle\psi_{a}, \hat{\mathrm{H}}_{a} \chi_{a}\right\rangle=\left\langle\hat{\mathrm{H}}_{a}^{\dagger} \psi_{a}, \chi_{a}\right\rangle$ is what essentially defines the adjoint operator $\hat{H}_{a}^{\dagger}$ on a vector space. If $\hat{H}_{a}$ is Hermitian (i.e., $\hat{\mathrm{H}}_{a}=\hat{\mathrm{H}}_{a}^{\dagger}$, and thus, $\hat{\mathrm{H}}_{a}^{\dagger}$ acts in the same way as $\hat{\mathrm{H}}_{a}$ ), then the relation $\left\langle\psi_{a}, \hat{\mathrm{H}}_{a} \chi_{a}\right\rangle=\left\langle\hat{\mathrm{H}}_{a} \psi_{a}, \chi_{a}\right\rangle$ is verified. If $\hat{\mathrm{H}}_{a}$ is self-adjoint in addition, then it must be Hermitian, but also, the domains of $\hat{\mathrm{H}}_{a}$ and its corresponding adjoint must be equal (for example, if $\chi_{a} \in \mathcal{D}\left(\hat{\mathrm{H}}_{a}\right)$ and $\psi_{a} \in \mathcal{D}\left(\hat{\mathrm{H}}_{a}^{\dagger}\right)$, then $\chi_{a}$ and $\psi_{a}$ must satisfy the same boundary condition).

Now, let us consider the following three particular cases, i.e., the following three particular Weyl Hamiltonian operators:

$$
\begin{equation*}
\hat{\mathrm{H}}_{a, j} \equiv-\mathrm{i} \hbar c(-1)^{a-1} \hat{\sigma}_{j} \frac{\partial}{\partial x^{j}}, \quad j=1,2,3 . \tag{17}
\end{equation*}
$$

Again, each of these operators is formally self-adjoint $\left(\hat{\mathrm{H}}_{a, j}=\hat{\mathrm{H}}_{a, j}^{\dagger}\right)$ and acts on the twocomponent wave functions $\varphi_{a, j}=\varphi_{a, j}\left(x^{j}, t\right)$ that belong to $\mathcal{H}=L^{2}(\Omega)^{2}$, where $\Omega \subset \mathbb{R}^{3}$ is a three-dimensional square box. However, in each case, the corresponding three-dimensional Weyl particle can only move inside an interval of size $\ell$ (i.e., inside a one-dimensional box) on the $x^{j}$-axis, with ends, for example, at $x^{j}=0$ and $x^{j}=\ell\left(\Rightarrow \Omega_{j}=[0, \ell]\right)$. The scalar product in $\mathcal{H}$ becomes $\left\langle\psi_{a, j}, \chi_{a, j}\right\rangle \equiv A_{j} \int_{\Omega_{j}} \mathrm{~d} x^{j} \psi_{a, j}^{\dagger} \chi_{a, j}$, where $A_{j}$ is the area of the side of the three-dimensional square box perpendicular to the one-dimensional interval, as expected. In this case, it can be demonstrated that the following relation is verified:

$$
\begin{equation*}
\left\langle\psi_{a, j}, \hat{\mathrm{H}}_{a, j} \chi_{a, j}\right\rangle=\left\langle\hat{\mathrm{H}}_{a, j} \psi_{a, j}, \chi_{a, j}\right\rangle-\left.\mathrm{i} \hbar c(-1)^{a-1} A_{j}\left[\psi_{a, j}^{\dagger} \hat{\sigma}_{j} \chi_{a, j}\right]\right|_{0} ^{\ell}, \tag{18}
\end{equation*}
$$

where $\left.[f]\right|_{0} ^{\ell} \equiv f\left(x^{j}=\ell, t\right)-f\left(x^{j}=0, t\right)$. Certainly, if the boundary conditions imposed on $\psi_{a, j}$ and $\chi_{a, j}$ at the endpoints of the interval $\Omega_{j}$ lead to cancellation of the boundary term in Eq. (18), then the operator $\hat{\mathrm{H}}_{a, j}$ will be at least Hermitian. Precisely, the most general family of self-adjoint boundary conditions for each of the Weyl operators $\hat{\mathrm{H}}_{a, j}(j=1,2,3)$ is
obtained below from the most general families of self-adjoint boundary conditions for three (free) Dirac operators in $(1+1)$ dimensions.

Similarly, the Dirac equation (9) in Hamiltonian form is

$$
\begin{equation*}
\mathrm{i} \hbar \hat{1}_{2} \frac{\partial}{\partial t} \Psi=\hat{\mathrm{h}} \Psi \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\mathrm{h}}=-\mathrm{i} \hbar c \hat{\gamma}^{0} \hat{\gamma}^{1} \frac{\partial}{\partial x} \tag{20}
\end{equation*}
$$

is the formally self-adjoint, or Hermitian, (free) Dirac Hamiltonian operator, $\hat{\mathrm{h}}=\hat{\mathrm{h}}^{\dagger}$. This operator acts on the two-component Dirac wave functions $\Psi=\Psi(x, t)$ that belong to the Hilbert space $\mathcal{H}=L^{2}([0, \ell])^{2}$. The scalar product in $\mathcal{H}$ is denoted by $\langle\Phi, \chi\rangle \equiv \int_{[0, \ell]} \mathrm{d} x \Phi^{\dagger} \chi$, and the norm is $\|\Phi\| \equiv \sqrt{\langle\Phi, \Phi\rangle}$, as usual. In this case, it can be demonstrated that the following relation is verified:

$$
\begin{equation*}
\langle\Phi, \hat{\mathrm{h}} \chi\rangle=\langle\hat{\mathrm{h}} \Phi, \chi\rangle-\left.\mathrm{i} \hbar c\left[\Phi^{\dagger} \hat{\gamma}^{0} \hat{\gamma}^{1} \chi\right]\right|_{0} ^{\ell} \tag{21}
\end{equation*}
$$

As we know, if the boundary conditions imposed on $\Phi$ and $\chi$ at the endpoints of the interval $[0, \ell]$ lead to cancellation of the boundary term in Eq. (21), then the operator $\hat{\mathrm{h}}$ will be at least Hermitian. However, given a set of boundary conditions imposed on $\chi \in \mathcal{D}(\hat{\mathrm{h}})$, if the cancellation of the boundary term in Eq. (21) only depends on imposing the same boundary conditions on $\Phi \in \mathcal{D}\left(\hat{\mathrm{h}}^{\dagger}\right)$, then the Hamiltonian is also a self-adjoint operator.

Now, with the operator $\hat{\mathrm{h}}$ in mind, let us construct the following three Dirac Hamiltonian operators:

$$
\begin{equation*}
\hat{\mathrm{h}}_{j} \equiv-\mathrm{i} \hbar c \hat{\sigma}_{j} \frac{\partial}{\partial x^{j}}, \quad j=1,2,3 \tag{22}
\end{equation*}
$$

Clearly, these three operators are essentially the three Weyl Hamiltonian operators $\hat{\mathrm{H}}_{a, j}$ $(j=1,2,3)$ in Eq. (17). Additionally, note that $\hat{\mathrm{h}}_{j}$ satisfies the relation given in Eq. (21) with the replacements $\hat{\mathrm{h}} \rightarrow \hat{\mathrm{h}}_{j}, \hat{\gamma}^{0} \hat{\gamma}^{1} \rightarrow \hat{\sigma}_{j}, \Phi \rightarrow \Phi_{j}$ and $\chi \rightarrow \chi_{j}$, and the latter relation is similar to the one that $\hat{\mathrm{H}}_{a, j}$ must satisfy (i.e., Eq. (18)). On the other hand, note that $\hat{\mathrm{h}}_{1}$ is the Dirac Hamiltonian for a one-dimensional Dirac particle in the interval $x \in[0, \ell]$ in both the Dirac and Majorana representations $\left(\hat{\gamma}^{0} \hat{\gamma}^{1}=\hat{\sigma}_{x}\right)$. Likewise, $\hat{h}_{2}$ is the Dirac Hamiltonian in the interval $y \in[0, \ell]$, in the Jackiw-Rebbi representation $\left(\hat{\gamma}^{0} \hat{\gamma}^{1}=\hat{\sigma}_{y}\right)$, and $\hat{\mathrm{h}}_{3}$ is the Dirac Hamiltonian in the interval $z \in[0, \ell]$ in the Weyl representation $\left(\hat{\gamma}^{0} \hat{\gamma}^{1}=\hat{\sigma}_{z}\right)$. Precisely, we already know which is the most general family of boundary conditions for the self-adjoint
operator $\hat{\mathrm{h}}_{3}$ (see Refs. [22, 23] and references therein), namely,

$$
\left[\begin{array}{l}
\varphi_{1}(z=\ell, t)  \tag{23}\\
\varphi_{2}(z=0, t)
\end{array}\right]=\hat{U}_{3}\left[\begin{array}{l}
\varphi_{2}(z=\ell, t) \\
\varphi_{1}(z=0, t)
\end{array}\right]
$$

where $\hat{U}_{3}$ is a unitary matrix; thus, this set of boundary conditions is characterized by four real parameters. Certainly, the latter result can be obtained using the von Neumann theory of self-adjoint extensions of symmetric operators, although the construction of the result can also be done using less rigorous arguments [22].

From the result in Eq. (23), we can find the most general set of boundary conditions for the self-adjoint operator $\hat{\mathrm{h}}_{1}$, which is written in both the Dirac and Majorana representations. For example, the Hamiltonian $\hat{\mathrm{h}}_{1}$ in the Dirac representation and $\hat{\mathrm{h}}_{3}$ (in the Weyl representation) are related via a unitary transformation, namely, $\hat{S}_{13}$; as a consequence, $\hat{\mathrm{h}}_{3}=\hat{S}_{13} \hat{\mathrm{~h}}_{1} \hat{S}_{13}^{\dagger}$ (certainly, we are assuming that the spatial variable is the same), and

$$
\left[\begin{array}{l}
\varphi_{1}  \tag{24}\\
\varphi_{2}
\end{array}\right]=\hat{S}_{13}\left[\begin{array}{l}
\varphi \\
\chi
\end{array}\right], \quad \text { with } \quad \hat{S}_{13}=\frac{1}{\sqrt{2}}\left(\hat{\sigma}_{x}+\hat{\sigma}_{z}\right)
$$

(see Ref. [14]). The first relation in Eq. (24) allows us to obtain the Dirac wave function associated with the operator $\hat{h}_{3}$ from the Dirac wave function associated with the operator $\hat{\mathrm{h}}_{1}$ in the Dirac representation. Thus, the most general family of boundary conditions for the self-adjoint operator $\hat{\mathrm{h}}_{1}$ is obtained by substituting the entire result in Eq. (24) into Eq. (23) and finally making the obvious replacements $z \rightarrow x$ and $\hat{U}_{3} \rightarrow \hat{U}_{1}$, namely,

$$
\left[\begin{array}{l}
\varphi(x=\ell, t)+\chi(x=\ell, t)  \tag{25}\\
\varphi(x=0, t)-\chi(x=0, t)
\end{array}\right]=\hat{U}_{1}\left[\begin{array}{l}
\varphi(x=\ell, t)-\chi(x=\ell, t) \\
\varphi(x=0, t)+\chi(x=0, t)
\end{array}\right]
$$

where $\hat{U}_{1}$ is the unitary matrix in this case. Certainly, in the Majorana representation, the most general family of boundary conditions for the self-adjoint operator $\hat{\mathrm{h}}_{1}$ has the same format as that given in Eq. (25), namely,

$$
\left[\begin{array}{l}
\phi_{1}(x=\ell, t)+\phi_{2}(x=\ell, t)  \tag{26}\\
\phi_{1}(x=0, t)-\phi_{2}(x=0, t)
\end{array}\right]=\hat{U}_{1}^{\prime}\left[\begin{array}{l}
\phi_{1}(x=\ell, t)-\phi_{2}(x=\ell, t) \\
\phi_{1}(x=0, t)+\phi_{2}(x=0, t)
\end{array}\right]
$$

where $\hat{U}_{1}^{\prime}$ is also a unitary matrix. To demonstrate this, we can follow the following simple procedure (which is identical to the one that led to the result given in Eq. (25)). Note, first,
that the Hamiltonian $\hat{\mathrm{h}}_{1}$ in the Majorana representation and $\hat{\mathrm{h}}_{3}$ (in the Weyl representation) are also related via a unitary transformation, namely, $\hat{S}_{13}^{\prime}$; therefore, $\hat{\mathrm{h}}_{3}=\hat{S}_{13}^{\prime} \hat{\mathrm{h}}_{1}\left(\hat{S}_{13}^{\prime}\right)^{\dagger}$, and

$$
\left[\begin{array}{l}
\varphi_{1}  \tag{27}\\
\varphi_{2}
\end{array}\right]=\hat{S}_{13}^{\prime}\left[\begin{array}{l}
\phi_{1} \\
\phi_{2}
\end{array}\right], \quad \text { with } \quad \hat{S}_{13}^{\prime}=\frac{1}{2}\left(-\mathrm{i} \hat{1}_{2}+\hat{\sigma}_{x}+\hat{\sigma}_{y}+\hat{\sigma}_{z}\right)=\exp \left(-\mathrm{i} \frac{\pi}{4}\right) \frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
\mathrm{i} & -\mathrm{i}
\end{array}\right]
$$

(see Ref. [14]). The first relation in Eq. (27) allows us to obtain the Dirac wave function associated with the operator $\hat{\mathrm{h}}_{3}$ from the Dirac wave function associated with the operator $\hat{\mathrm{h}}_{1}$ in the Majorana representation. Then, the most general family of boundary conditions for the self-adjoint operator $\hat{\mathrm{h}}_{1}$, i.e., Eq. (26), is obtained by substituting the entire result in Eq. (27) into Eq. (23), also making the obvious replacements $z \rightarrow x$ and $\hat{U}_{3} \rightarrow \hat{U}_{1}^{\prime}$ and, finally, certain simplifications.

Similarly, from the result in Eq. (23), we can find the most general set of boundary conditions for the self-adjoint operator $\hat{\mathrm{h}}_{2}$, which is written in the Jackiw-Rebbi representation. In effect, the Hamiltonians $\hat{\mathrm{h}}_{2}$ and $\hat{\mathrm{h}}_{3}$ (in the Weyl representation) are related via the unitary matrix $\hat{S}_{23}$; thus, $\hat{\mathrm{h}}_{3}=\hat{S}_{23} \hat{\mathrm{~h}}_{2} \hat{S}_{23}^{\dagger}$, and

$$
\left[\begin{array}{l}
\varphi_{1}  \tag{28}\\
\varphi_{2}
\end{array}\right]=\hat{S}_{23}\left[\begin{array}{l}
\chi_{1} \\
\chi_{2}
\end{array}\right], \quad \text { with } \quad \hat{S}_{23}=\frac{1}{\sqrt{2}}\left(\hat{1}_{2}-\mathrm{i} \hat{\sigma}_{x}\right)
$$

The first relation in Eq. (28) allows us to obtain the Dirac wave function associated with the operator $\hat{\mathrm{h}}_{3}$ from the Dirac wave function associated with the operator $\hat{\mathrm{h}}_{2}$. Thus, the most general family of boundary conditions for the self-adjoint operator $\hat{h}_{2}$ is obtained by substituting the entire result in Eq. (28) into Eq. (23) and finally making the obvious replacements $z \rightarrow y$ and $\hat{U}_{3} \rightarrow \hat{U}_{2}$, namely,

$$
\left[\begin{array}{l}
\chi_{1}(y=\ell, t)-\mathrm{i} \chi_{2}(y=\ell, t)  \tag{29}\\
\chi_{2}(y=0, t)-\mathrm{i} \chi_{1}(y=0, t)
\end{array}\right]=\hat{U}_{2}\left[\begin{array}{l}
\chi_{2}(y=\ell, t)-\mathrm{i} \chi_{1}(y=\ell, t) \\
\chi_{1}(y=0, t)-\mathrm{i} \chi_{2}(y=0, t)
\end{array}\right]
$$

where $\hat{U}_{2}$ is a unitary matrix.
From the results given in Eqs. (23), (25) (or (26)) and (29), we can immediately write the most general families of boundary conditions for the self-adjoint Weyl operators $\hat{\mathrm{H}}_{a, 3}$, $\hat{\mathrm{H}}_{a, 1}$ and $\hat{\mathrm{H}}_{a, 2}$, respectively. First, the operator

$$
\begin{equation*}
\hat{\mathrm{H}}_{a, 1} \equiv-\mathrm{i} \hbar c(-1)^{a-1} \hat{\sigma}_{x} \frac{\partial}{\partial x} \quad(a=1,2) \tag{30}
\end{equation*}
$$

can act on two-component wave functions $\varphi_{a, 1} \equiv \varphi_{a}=\left[\varphi_{a}^{\mathrm{t}} \varphi_{a}^{\mathrm{b}}\right]^{\mathrm{T}}$ that satisfy any of the following boundary conditions:

$$
\left[\begin{array}{l}
\varphi_{a}^{\mathrm{t}}(x=\ell, t)+\varphi_{a}^{\mathrm{b}}(x=\ell, t)  \tag{31}\\
\varphi_{a}^{\mathrm{t}}(x=0, t)-\varphi_{a}^{\mathrm{b}}(x=0, t)
\end{array}\right]=\hat{A}_{1}\left[\begin{array}{l}
\varphi_{a}^{\mathrm{t}}(x=\ell, t)-\varphi_{a}^{\mathrm{b}}(x=\ell, t) \\
\varphi_{a}^{\mathrm{t}}(x=0, t)+\varphi_{a}^{\mathrm{b}}(x=0, t)
\end{array}\right]
$$

where $\hat{A}_{1}$ is a unitary matrix. Because we want to highlight here the dependence of the boundary conditions with the label $a$, we can assume that the operator

$$
\begin{equation*}
\hat{\mathrm{H}}_{a, 2} \equiv-\mathrm{i} \hbar c(-1)^{a-1} \hat{\sigma}_{y} \frac{\partial}{\partial y} \quad(a=1,2) \tag{32}
\end{equation*}
$$

can act on two-component wave functions that we write again as $\varphi_{a, 2} \equiv \varphi_{a}=\left[\varphi_{a}^{\mathrm{t}} \varphi_{a}^{\mathrm{b}}\right]^{\mathrm{T}}$ and that satisfy any of the following boundary conditions:

$$
\left[\begin{array}{c}
\varphi_{a}^{\mathrm{t}}(y=\ell, t)-\mathrm{i} \varphi_{a}^{\mathrm{b}}(y=\ell, t)  \tag{33}\\
\varphi_{a}^{\mathrm{b}}(y=0, t)-\mathrm{i} \varphi_{a}^{\mathrm{t}}(y=0, t)
\end{array}\right]=\hat{A}_{2}\left[\begin{array}{l}
\varphi_{a}^{\mathrm{b}}(y=\ell, t)-\mathrm{i} \varphi_{a}^{\mathrm{t}}(y=\ell, t) \\
\varphi_{a}^{\mathrm{t}}(y=0, t)-\mathrm{i} \varphi_{a}^{\mathrm{b}}(y=0, t)
\end{array}\right]
$$

where $\hat{A}_{2}$ is a unitary matrix. Similarly, the operator

$$
\begin{equation*}
\hat{\mathrm{H}}_{a, 3} \equiv-\mathrm{i} \hbar c(-1)^{a-1} \hat{\sigma}_{z} \frac{\partial}{\partial z} \quad(a=1,2) \tag{34}
\end{equation*}
$$

acts on the wave functions $\varphi_{a, 3} \equiv \varphi_{a}=\left[\varphi_{a}^{\mathrm{t}} \varphi_{a}^{\mathrm{b}}\right]^{\mathrm{T}}$ that satisfy at least one of the following infinite boundary conditions:

$$
\left[\begin{array}{l}
\varphi_{a}^{\mathrm{t}}(z=\ell, t)  \tag{35}\\
\varphi_{a}^{\mathrm{b}}(z=0, t)
\end{array}\right]=\hat{A}_{3}\left[\begin{array}{l}
\varphi_{a}^{\mathrm{b}}(z=\ell, t) \\
\varphi_{a}^{\mathrm{t}}(z=0, t)
\end{array}\right]
$$

where $\hat{A}_{3}$ is a unitary matrix.
Note that the Weyl equations with the operators $\hat{\mathrm{H}}_{a, 1}$ and $\hat{\mathrm{H}}_{a, 3}$, i.e., Eq. (14) with the replacements $\hat{\mathrm{H}}_{a} \rightarrow \hat{\mathrm{H}}_{a, 1}$ and $\hat{\mathrm{H}}_{a} \rightarrow \hat{\mathrm{H}}_{a, 3}$, can provide real solutions (see the comment made in the last paragraph of section II, subsection A). Thus, if we impose on the corresponding wave function $\varphi_{a}$ the reality condition $\left(\varphi_{a}=\varphi_{a}^{*}\right), \varphi_{a}$ and $\varphi_{a}^{*}$ must meet the boundary conditions in Eqs. (31) and (35), in which case the unitary matrices in these equations, $\hat{A}_{1}$ and $\hat{A}_{3}$, must also be real, i.e., these matrices must be orthogonal. Consequently, in this case, the families of general boundary conditions given in Eqs. (31) and (35) only depend on one real parameter. On the other hand, the Weyl equations in (14) with the replacement $\hat{\mathrm{H}}_{a} \rightarrow \hat{\mathrm{H}}_{a, 2}$ cannot give real-valued solutions. Thus, these necessarily complex solutions support any boundary condition included in the real four-parameter general family of boundary conditions given in Eq. (33) (see appendix, subsection C).

## IV. BOUNDARY CONDITIONS FOR THE 1D WEYL PARTICLE IN A (1D) BOX

We have described above general families of self-adjoint boundary conditions for a 3D Weyl particle in three one-dimensional boxes. Let us now consider a one-dimensional Weyl particle in a box of size $\ell$, with ends, for example, at $x=0$ and $x=\ell$. First, the two Weyl equations in Eq. (11) can be written in a single equation in their canonical form as follows:

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial}{\partial t} \varphi_{a}=\hat{\mathrm{h}}_{a} \varphi_{a}, \quad a=1,2 \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\mathrm{h}}_{a} \equiv-\mathrm{i} \hbar c(-1)^{a-1} \frac{\partial}{\partial x} \tag{37}
\end{equation*}
$$

is the formally self-adjoint, or Hermitian, one-dimensional Weyl Hamiltonian operator, i.e., $\hat{\mathrm{h}}_{a}=\hat{\mathrm{h}}_{a}^{\dagger}$. Note that $\hat{\mathrm{h}}_{a}$ is practically the non-relativistic momentum operator [24], i.e., $\hat{\mathrm{h}}_{a}=c(-1)^{a-1} \hat{\mathrm{p}}_{[1]}$ (as we know, the label $a$ indicates the type of 1D Weyl particle that we are describing). The Hamiltonian is also a self-adjoint operator; this is (essentially) because its domain, i.e., the set of Weyl one-component wave functions $\varphi_{a}=\varphi_{a}(x, t)$ in the Hilbert space of the square integrable functions $\mathcal{H}=L^{2}[0, \ell]$ on which $\hat{\mathrm{h}}_{a}$ can act $\left(\equiv \mathcal{D}\left(\hat{\mathrm{h}}_{a}\right) \subset \mathcal{H}\right)$, includes the generalized periodic boundary condition dependent on a single parameter, namely,

$$
\begin{equation*}
\varphi_{a}(\ell, t)=\exp (\mathrm{i} \eta) \varphi_{a}(0, t) \tag{38}
\end{equation*}
$$

with $\eta \in\left[0,2 \pi\right.$ ); in addition, $\hat{\mathrm{h}}_{a} \varphi_{a} \in \mathcal{H}$ (for a complete derivation of the latter result, see Ref. [24]). Moreover, the scalar product in $\mathcal{H}$ is denoted by $\left\langle\psi_{a}, \chi_{a}\right\rangle \equiv \int_{0}^{\ell} \mathrm{d} x \psi_{a}^{*} \chi_{a}$, and the norm is $\left\|\psi_{a}\right\| \equiv \sqrt{\left\langle\psi_{a}, \psi_{a}\right\rangle}$. Precisely, $\hat{\mathrm{h}}_{a}$ satisfies the following condition, that is, the hermiticity condition (or, in this case, the self-adjointness condition):

$$
\begin{equation*}
\left\langle\psi_{a}, \hat{\mathrm{~h}}_{a} \chi_{a}\right\rangle=\left\langle\hat{\mathrm{h}}_{a} \psi_{a}, \chi_{a}\right\rangle-\left.\mathrm{i} \hbar c(-1)^{a-1}\left[\psi_{a}^{*} \chi_{a}\right]\right|_{0} ^{\ell}=\left\langle\hat{\mathrm{h}}_{a} \psi_{a}, \chi_{a}\right\rangle, \tag{39}
\end{equation*}
$$

where $\psi_{a}$ and $\chi_{a}$ are Weyl wave functions in $\mathcal{D}\left(\hat{\mathrm{h}}_{a}\right)=\mathcal{D}\left(\hat{\mathrm{h}}_{a}^{\dagger}\right)$.
Note that the (free) Weyl equations in $(1+1)$ dimensions (see Eq. (36)) can always provide real-valued solutions. Thus, if we impose on the wave function $\varphi_{a}$ the reality condition, i.e., $\varphi_{a}=\varphi_{a}^{*}$, then $\varphi_{a}$ and $\varphi_{a}^{*}$ must satisfy the same boundary condition written above, in which case the phase factor $\exp (\mathrm{i} \eta)$ must be real. The latter condition implies that $\eta=0$ or $\eta=\pi$. Thus, the boundary conditions for the (uncharged) 1D (free) Weyl particle are
$\varphi_{a}(\ell, t)=\varphi_{a}(0, t)$, i.e., the periodic boundary condition, and $\varphi_{a}(\ell, t)=-\varphi_{a}(0, t)$, i.e., the antiperiodic boundary condition.

## V. CONCLUSIONS

Our primary objective in this article was to obtain the most general families of (self-adjoint) boundary conditions that can be imposed on the general solutions of the time-dependent Weyl equations that describe a 3D Weyl particle and a 1D Weyl particle in a one-dimensional box. Because the one-dimensional box can be placed on any Cartesian axis, one has three Weyl equations for the 3D Weyl particle in the box (i.e., Eq. (14) with the replacement $\hat{\mathrm{H}}_{a} \rightarrow \hat{\mathrm{H}}_{a, j}$, where $\hat{\mathrm{H}}_{a, j}$ is given in Eq. (17)). Each of the Weyl Hamiltonians present in these equations can be identified with a Dirac Hamiltonian that describes a 1D Dirac particle in a one-dimensional box. Because we know which is the most general family of boundary conditions for the Dirac operator in the Weyl representation, we were able to construct, from the latter, the most general families of boundary conditions for the other two Dirac operators by means of unitary transformations (i.e., via changes of representation). In the end, the three general families of (self-adjoint) boundary conditions for the 3D Weyl particle in the 1D box are characterized by four real parameters (which, in each case, constitute a $2 \times 2$ unitary matrix). In the cases where Weyl's equations can give real-valued general solutions, each family of four real parameters becomes two families each characterized by a single real parameter (each parameter within a $2 \times 2$ orthogonal matrix).

On the other hand, for the 1D Weyl particle, we have a Weyl equation whose Weyl Hamiltonian is similar to the non-relativistic momentum operator for the particle in a box (see Eqs. (36) and (37)). Thus, the most general family of (self-adjoint) boundary conditions for the 1D Weyl particle in the (1D) box is characterized by one real parameter. In this case, the general solutions of the Weyl equation can always be real-valued, but then these solutions can only accept periodic and antiperiodic boundary conditions.

The mathematical manner in which the Weyl Hamiltonian (and that of Dirac) acts on a wave function in $(3+1)$ dimensions, that is, in a way that depends on the direction in which the particles move and on certain matrices along that direction (the " $\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}_{[2]}$ " term in the Weyl Hamiltonian), has found applications that far surpass high-energy physics. For example, in condensed-matter physics, the Weyl and the Dirac equations can be used to
describe the band structure of Dirac materials (or systems where the low-energy electronic excitations are essentially described by Weyl or Dirac equations). A famous two-dimensional Dirac material is graphene, which hosts excitations described by a 2D Weyl equation (see, for example, Refs. [25, 26] and references therein). We hope that our results, although valid for a one-dimensional system, can also be applied to some of the various accessible systems within condensed-matter physics.

## VI. APPENDIX

## A. On rotations

Let us write an ordinary spatial rotation through an angle $\theta$ around the $x^{j}$-axis, that is, $\left[c t^{\prime} x^{\prime} y^{\prime} z^{\prime}\right]^{\mathrm{T}}=\hat{\Lambda}_{j}[c t x y z]^{\mathrm{T}}$ (i.e., $x^{\mu \prime}=\left(\Lambda_{j}\right)^{\mu}{ }_{\nu} x^{\nu} \Rightarrow x^{\mu}=\left(\Lambda_{j}^{-1}\right)^{\mu}{ }_{\nu} x^{\nu}{ }^{\prime}$ ), where

$$
\begin{equation*}
\hat{\Lambda}_{j}=\hat{\Lambda}_{j}(\theta)=\exp \left(\mathrm{i} \theta \hat{J}_{j}\right) \tag{A1}
\end{equation*}
$$

with
$\hat{J}_{1}=\left[\begin{array}{ll}\hat{0}_{2} & \hat{0}_{2} \\ \hat{0}_{2} & \hat{\sigma}_{y}\end{array}\right], \quad \hat{J}_{2}=\left[\begin{array}{cc}\hat{0}_{2} & \frac{\mathrm{i}}{2}\left(\hat{1}_{2}-\hat{\sigma}_{z}\right) \\ -\frac{\mathrm{i}}{2}\left(\hat{1}_{2}-\hat{\sigma}_{z}\right) & \hat{0}_{2}\end{array}\right], \quad \hat{J}_{3}=\left[\begin{array}{cc}\hat{0}_{2} & -\frac{\mathrm{i}}{2}\left(\hat{\sigma}_{x}-\mathrm{i} \hat{\sigma}_{y}\right) \\ \frac{\mathrm{i}}{2}\left(\hat{\sigma}_{x}+\mathrm{i} \hat{\sigma}_{y}\right) & \hat{0}_{2}\end{array}\right]$.
Then, under this linear transformation, the Dirac wave function in $(3+1)$ dimensions transforms as $\Psi^{\prime}\left(x^{k \prime}, t^{\prime}\right)=\hat{S}\left(\Lambda_{j}\right) \Psi\left(x^{k}, t\right)(k=1,2,3)$, where $\hat{S}\left(\Lambda_{j}\right)$, which obeys the relation $\left(\Lambda_{j}\right)^{\mu}{ }_{\nu} \hat{\gamma}^{\nu}=\hat{S}^{-1}\left(\Lambda_{j}\right) \hat{\gamma}^{\mu} \hat{S}\left(\Lambda_{j}\right)$, is given by $\hat{S}\left(\Lambda_{j}\right)=\exp \left(\mathrm{i} \theta \mathrm{i} \hat{\gamma}^{k} \hat{\gamma}^{l} / 2\right)(l=1,2,3)$ for cyclic $\{j, k, l\}$. Because $\hat{\boldsymbol{\Sigma}}=\left(\mathrm{i} \hat{\gamma}^{2} \hat{\gamma}^{3}, \mathrm{i} \hat{\gamma}^{3} \hat{\gamma}^{1}, \mathrm{i} \hat{\gamma}^{1} \hat{\gamma}^{2}\right)\left(=\hat{\gamma}^{5} \hat{\gamma}^{0} \hat{\gamma}\right)$, we can write the $4 \times 4$ matrix $\hat{S}\left(\Lambda_{j}\right)$ as follows:

$$
\begin{equation*}
\hat{S}\left(\Lambda_{j}\right)=\exp \left(\mathrm{i} \theta \frac{\hat{\Sigma}^{j}}{2}\right) \tag{A3}
\end{equation*}
$$

i.e., the spin operator in $(3+1)$ dimensions $\hat{\mathbf{S}}=\hbar \hat{\boldsymbol{\Sigma}} / 2$ is essentially the generator of spatial rotations. In the Weyl representation, $\hat{S}\left(\Lambda_{j}\right)$ is a block-diagonal matrix because $\hat{\boldsymbol{\Sigma}}=\operatorname{diag}(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\sigma}})$, and we obtain the following results:

$$
\left[\begin{array}{l}
\varphi_{1}^{\mathrm{t}}\left(x^{j \prime}, t^{\prime}\right)  \tag{A4}\\
\varphi_{1}^{\mathrm{b} \prime}\left(x^{j \prime}, t^{\prime}\right)
\end{array}\right]=\left[\cos \left(\frac{\theta}{2}\right) \hat{1}_{2}+\mathrm{i} \sin \left(\frac{\theta}{2}\right) \hat{\sigma}_{j}\right]\left[\begin{array}{l}
\varphi_{1}^{\mathrm{t}}\left(x^{j}, t\right) \\
\varphi_{1}^{\mathrm{b}}\left(x^{j}, t\right)
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
\varphi_{2}^{\mathrm{t} \prime}\left(x^{j \prime}, t^{\prime}\right) \\
\varphi_{2}^{\mathrm{b} \prime}\left(x^{j \prime}, t^{\prime}\right)
\end{array}\right]=\left[\cos \left(\frac{\theta}{2}\right) \hat{1}_{2}+\mathrm{i} \sin \left(\frac{\theta}{2}\right) \hat{\sigma}_{j}\right]\left[\begin{array}{l}
\varphi_{2}^{\mathrm{t}}\left(x^{j}, t\right) \\
\varphi_{2}^{\mathrm{b}}\left(x^{j}, t\right)
\end{array}\right] .
$$

That is, the two-component wave functions (or Weyl spinors) in (3+1) dimensions $\varphi_{1}$ and $\varphi_{2}$ transform in the same way under spatial rotations. Obviously, in (1+1) dimensions, a pure (spatial) rotation is not possible.

## B. On the concept of helicity

In $(3+1)$ dimensions, the eigenstates $\left(\Psi_{+}\right)_{\mathbf{p}}=\left[\left(\varphi_{1}\right)_{\mathbf{p}} 0\right]^{\mathrm{T}}$ and $\left(\Psi_{-}\right)_{\mathbf{p}}=\left[0\left(\varphi_{2}\right)_{\mathbf{p}}\right]^{\mathrm{T}}$ of the helicity operator $\hat{\lambda} \equiv \hat{\boldsymbol{\Sigma}} \cdot \hat{\mathbf{p}} /\|\mathbf{p}\|=\hat{\mathbf{S}} \cdot \hat{\mathbf{p}} / \frac{\hbar}{2}\|\mathbf{p}\|\left(=\operatorname{diag}\left(\hat{\lambda}_{[2]}, \hat{\lambda}_{[2]}\right)\right)$ satisfy the following relations that depend on the sign of the energy:

$$
\begin{equation*}
\hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{p}}}{\|\mathbf{p}\|}\left(\Psi_{+}\right)_{\mathbf{p}}=\operatorname{sgn}(E) \frac{\hbar}{2} \hat{1}_{4}\left(\Psi_{+}\right)_{\mathbf{p}}, \quad \hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{p}}}{\|\mathbf{p}\|}\left(\Psi_{-}\right)_{\mathbf{p}}=-\operatorname{sgn}(E) \frac{\hbar}{2} \hat{1}_{4}\left(\Psi_{-}\right)_{\mathbf{p}} \tag{B1}
\end{equation*}
$$

and therefore,

$$
\begin{equation*}
\hat{\mathbf{S}}_{[2]} \cdot \frac{\hat{\mathbf{p}}_{[2]}}{\left\|\mathbf{p}_{[2]}\right\|}\left(\varphi_{1}\right)_{\mathbf{p}}=\operatorname{sgn}(E) \frac{\hbar}{2} \hat{1}_{2}\left(\varphi_{1}\right)_{\mathbf{p}}, \quad \hat{\mathbf{S}}_{[2]} \cdot \frac{\hat{\mathbf{p}}_{[2]}}{\left\|\mathbf{p}_{[2]}\right\|}\left(\varphi_{2}\right)_{\mathbf{p}}=-\operatorname{sgn}(E) \frac{\hbar}{2} \hat{1}_{2}\left(\varphi_{2}\right)_{\mathbf{p}} \tag{B2}
\end{equation*}
$$

where $\hat{\lambda}_{[2]} \equiv \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|=\hat{\mathbf{S}}_{[2]} \cdot \hat{\mathbf{p}}_{[2]} / \frac{\hbar}{2}\left\|\mathbf{p}_{[2]}\right\|$ and $\left\|\mathbf{p}_{[2]}\right\|=\|\mathbf{p}\|$. Thus, the eigenvalues of the operators $\hat{\lambda}$ and $\hat{\lambda}_{[2]}$ only indicate whether the direction of the spin of the particle in question is parallel or antiparallel to its respective momentum; however, all of these eigenvalues are also dependent on the sign of the energy.

Let us now introduce the so-called (Hermitian) classical velocity operator $\hat{\mathrm{v}}_{\mathrm{cl}} \equiv c^{2} \hat{\mathbf{p}} \hat{\mathrm{E}}^{-1}$ (which corresponds to the formula of classical relativistic mechanics that provides the velocity as a function of momentum and energy), where $\hat{E}$ is the Dirac Hamiltonian operator [27]. Clearly, if $\hat{\mathbf{v}}_{\mathrm{cl}}$ acts on the Dirac plane-wave solution $\Psi_{\mathbf{p}}$, one obtains the eigenvalue $\mathbf{v}_{\mathrm{cl}}=c^{2} \mathbf{p} / E$, i.e., $\mathbf{v}_{\mathrm{cl}}=\operatorname{sgn}(E) c \mathbf{p} /\|\mathbf{p}\|\left(\Rightarrow\left\|\mathbf{v}_{\mathrm{cl}}\right\|=c\right.$, as expected). Then, we can use these results to write the relations in (B1) and (B2) such that they do not depend on the sign of the energy, that is,

$$
\begin{equation*}
\hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{v}}_{\mathrm{cl}}}{c}\left(\Psi_{+}\right)_{\mathbf{p}}=\frac{\hbar}{2} \hat{1}_{4}\left(\Psi_{+}\right)_{\mathbf{p}}, \quad \hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{v}}_{\mathrm{cl}}}{c}\left(\Psi_{-}\right)_{\mathbf{p}}=-\frac{\hbar}{2} \hat{1}_{4}\left(\Psi_{-}\right)_{\mathbf{p}}, \tag{B3}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{S}}_{[2]} \cdot \frac{\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]}}{c}\left(\varphi_{1}\right)_{\mathbf{p}}=\frac{\hbar}{2} \hat{1}_{2}\left(\varphi_{1}\right)_{\mathbf{p}}, \quad \hat{\mathbf{S}}_{[2]} \cdot \frac{\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]}}{c}\left(\varphi_{2}\right)_{\mathbf{p}}=-\frac{\hbar}{2} \hat{1}_{2}\left(\varphi_{2}\right)_{\mathbf{p}} \tag{B4}
\end{equation*}
$$

respectively $\left(\right.$ where $\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]}=\operatorname{sgn}(E) c \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|$ and $\left.\hat{\mathbf{v}}_{\mathrm{cl}}=\operatorname{diag}\left(\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]},\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]}\right)\right)$. In this way, the eigenvalues of the operators $\hat{\mathbf{S}} \cdot \hat{\mathbf{v}}_{\mathrm{cl}} / c$ and $\hat{\mathbf{S}}_{[2]} \cdot\left(\hat{\mathbf{v}}_{\mathrm{cl}}\right)_{[2]} / c$ indicate whether the direction of the spin of the particle in question is parallel or antiparallel to the movement of the particle. For example, the spin of the 3D Weyl particle described by $\left(\varphi_{1}\right)_{\mathbf{p}}$ is always parallel to its direction of motion, but the spin of the 3 D Weyl particle described by $\left(\varphi_{2}\right)_{\mathbf{p}}$ is always antiparallel to its direction of motion.

As we have seen, the eigenstates of the operator $\hat{p} /|\mathrm{p}|$ in $(1+1)$ dimensions, $\left(\Psi_{+}\right)_{\mathrm{p}}=$ $\left[\left(\varphi_{1}\right)_{\mathrm{p}} 0\right]^{\mathrm{T}}$ and $\left(\Psi_{-}\right)_{\mathrm{p}}=\left[0\left(\varphi_{2}\right)_{\mathrm{p}}\right]^{\mathrm{T}}$, comply with relations that depend on the sign of the energy, namely,

$$
\begin{equation*}
\frac{\hat{\mathrm{p}}}{|\mathrm{p}|}\left(\Psi_{+}\right)_{\mathrm{p}}=\operatorname{sgn}(E) \hat{1}_{2}\left(\Psi_{+}\right)_{\mathrm{p}}, \quad \frac{\hat{\mathrm{p}}}{|\mathrm{p}|}\left(\Psi_{-}\right)_{\mathrm{p}}=-\operatorname{sgn}(E) \hat{1}_{2}\left(\Psi_{-}\right)_{\mathrm{p}} \tag{B5}
\end{equation*}
$$

from which similar relations for $\left(\varphi_{1}\right)_{\mathrm{p}}$ and $\left(\varphi_{2}\right)_{\mathrm{p}}$ are immediately obtained, namely,

$$
\begin{equation*}
\frac{\hat{\mathrm{p}}_{[1]}}{\left|\mathrm{p}_{[1]}\right|}\left(\varphi_{1}\right)_{\mathrm{p}}=\operatorname{sgn}(E)\left(\varphi_{1}\right)_{\mathrm{p}}, \quad \frac{\hat{\mathrm{p}}_{[1]}}{\left|\mathrm{p}_{[1]}\right|}\left(\varphi_{2}\right)_{\mathrm{p}}=-\operatorname{sgn}(E)\left(\varphi_{2}\right)_{\mathrm{p}}, \tag{B6}
\end{equation*}
$$

where $\hat{\mathrm{p}} /|\mathrm{p}|=\hat{\mathrm{p}}_{[1]} \hat{1}_{2} /\left|\mathrm{p}_{[1]}\right|$ and $\left|\mathrm{p}_{[1]}\right|=|\mathrm{p}|$. Clearly, the operators $\hat{\mathrm{p}} /|\mathrm{p}|$ and $\hat{\mathbf{S}} \cdot \hat{\mathbf{p}} /\|\mathbf{p}\|$, as well as $\hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|$ and $\hat{\mathbf{S}}_{[2]} \cdot \hat{\mathbf{p}}_{[2]} /\left\|\mathbf{p}_{[2]}\right\|$, have a certain similarity (when acting on their respective chiral plane-wave eigenstates). The Dirac plane-wave $\Psi_{\mathrm{p}}$ is also an eigensolution of the (Hermitian) classical velocity operator $\hat{\mathrm{v}}_{\mathrm{cl}} \equiv c^{2} \hat{\mathrm{p}} \hat{\mathrm{E}}^{-1}$ and has eigenvalue $\mathrm{v}_{\mathrm{cl}}=c^{2} \mathrm{p} / E=$ $\operatorname{sgn}(E) c \mathrm{p} /|\mathrm{p}|(\hat{\mathrm{E}}(=\hat{\mathrm{h}})$ is the Dirac Hamiltonian operator in $(1+1)$ dimensions). This fact allows us to write the relations in (B5) and (B6) in a form independent of the energy sign, namely,

$$
\begin{equation*}
\frac{\hat{\mathrm{v}}_{\mathrm{cl}}}{c}\left(\Psi_{+}\right)_{\mathrm{p}}=\hat{1}_{2}\left(\Psi_{+}\right)_{\mathrm{p}}, \quad \frac{\hat{\mathrm{v}}_{\mathrm{cl}}}{c}\left(\Psi_{-}\right)_{\mathrm{p}}=-\hat{1}_{2}\left(\Psi_{-}\right)_{\mathrm{p}} \tag{B7}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\left(\hat{\mathrm{v}}_{\mathrm{cl}}\right)_{[1]}}{c}\left(\varphi_{1}\right)_{\mathrm{p}}=\left(\varphi_{1}\right)_{\mathrm{p}}, \quad \frac{\left(\hat{\mathrm{v}}_{\mathrm{cl}}\right)_{[1]}}{c}\left(\varphi_{2}\right)_{\mathrm{p}}=-\left(\varphi_{2}\right)_{\mathrm{p}} \tag{B8}
\end{equation*}
$$

respectively (where $\left(\hat{\mathrm{v}}_{\mathrm{cl}}\right)_{[1]}=\operatorname{sgn}(E) c \hat{\mathrm{p}}_{[1]} /\left|\mathrm{p}_{[1]}\right|$ and $\hat{\mathrm{v}}_{\mathrm{cl}}=\left(\hat{\mathrm{v}}_{\mathrm{cl}}{ }_{[1]} \hat{1}_{2}\right.$ ). Clearly, the eigenvalues of the operators $\hat{\mathrm{v}}_{\mathrm{cl}} / c$ and $\left(\hat{\mathrm{v}}_{\mathrm{cl}}\right)_{[1]} / c$ indicate whether the particle in question, whether it is a 1D Dirac particle or a 1D Weyl particle, actually moves to the right or to the left. For example, the 1D Weyl particle described by $\left(\varphi_{1}\right)_{\mathrm{p}}$ always moves to the right (left), but the 1D Weyl particle described by $\left(\varphi_{2}\right)_{\mathrm{p}}$ moves to the left (right).

## C. On the boundary conditions for the Weyl equations

We have obtained the most general families of boundary conditions for the (time-dependent) Weyl equations given in Eq. (14) (i.e., in (3+1) dimensions), where the (self-adjoint) Weyl Hamiltonian operators present are precisely the operators $\hat{\mathrm{H}}_{a, j}$ given in Eq. (17). Each of the three families of boundary conditions (labeled by $j=1,2,3$ and given in Eqs. (31), (33) and (35)) is parametrized by a unitary $2 \times 2$ matrix, that is, by $2^{2}=4$ real parameters. A feasible parametrization for these unitary matrices, for example, for the matrix $\hat{A}_{1}$ in Eq. (31), is given by

$$
\hat{A}_{1}=\exp (\mathrm{i} \mu)\left[\begin{array}{cc}
m_{0}-\mathrm{i} m_{3} & -m_{2}-\mathrm{i} m_{1}  \tag{C1}\\
m_{2}-\mathrm{i} m_{1} & m_{0}+\mathrm{i} m_{3}
\end{array}\right]
$$

where $\mu \in[0, \pi)$, and real quantities $m_{0}, m_{1}, m_{2}$ and $m_{3}$, satisfy $\left(m_{0}\right)^{2}+\left(m_{1}\right)^{2}+\left(m_{2}\right)^{2}+$ $\left(m_{3}\right)^{2}=1$ (but also $\operatorname{det}\left(\hat{A}_{1}\right)=\exp (\mathrm{i} 2 \mu)$ ) [28]. For other interesting examples of Hamiltonians operators whose self-adjoint extensions (or sets of general boundary conditions) are characterized in terms of unitary matrices, see Refs. [29, 30].

On the other hand, all boundary conditions that are part of each of these three families of self-adjoint boundary conditions cancel the boundary term in Eq. (18), which implies that

$$
\begin{equation*}
\left.\left.c\left[\varphi_{a, j}^{\dagger} \hat{\sigma}_{j} \varphi_{a, j}\right]\right|_{0} ^{\ell} \equiv\left[J_{a, j}\right]\right|_{0} ^{\ell}=0 \quad \Rightarrow \quad J_{a, j}\left(x^{j}=\ell, t\right)=J_{a, j}\left(x^{j}=0, t\right) \tag{C2}
\end{equation*}
$$

where $J_{a, j}=J_{a, j}\left(x^{j}, t\right)$ is the probability current density [11. Thus, all of the self-adjoint boundary conditions lead to the equality of $J_{a, j}$ at the ends of the box. Within each general family of boundary conditions, there are boundary conditions that simply cancel the probability current density at these extremes; they are called confining boundary conditions. For example, the following confining boundary conditions for the Weyl Hamiltonian $\hat{\mathrm{H}}_{a, 1}$ are contained in Eq. (31): $\varphi_{a}^{\mathrm{t}}(x=\ell, t)=\varphi_{a}^{\mathrm{t}}(x=0, t)=0\left(\hat{A}_{1}=-\hat{1}_{2}\right)$, i.e., the upper component of the wave function $\varphi_{a, 1} \equiv \varphi_{a}$ can satisfy the Dirichlet boundary condition; $\varphi_{a}^{\mathrm{b}}(x=\ell, t)=\varphi_{a}^{\mathrm{b}}(x=0, t)=0\left(\hat{A}_{1}=+\hat{1}_{2}\right)$, i.e., the lower component of the wave function $\varphi_{a, 1} \equiv \varphi_{a}$ can also satisfy the Dirichlet boundary condition. However, the entire two-component Weyl wave function $\varphi_{a, 1} \equiv \varphi_{a}$ does not support this boundary condition at the walls of the box, i.e., the latter is not contained in Eq. (31). This result is also fulfilled by the Dirac wave function [31. Likewise, there are also boundary conditions that do not cancel $J_{a, j}$ at the ends of the box; they are called non-confining boundary conditions. For
example, the following non-confining boundary conditions for the Weyl Hamiltonian $\hat{\mathrm{H}}_{a, 1}$ are also contained in Eq. (31): $\varphi_{a}^{\mathrm{t}}(x=\ell, t)=\varphi_{a}^{\mathrm{t}}(x=0, t)$ and $\varphi_{a}^{\mathrm{b}}(x=\ell, t)=\varphi_{a}^{\mathrm{b}}(x=0, t)$ ( $\hat{A}_{1}=+\hat{\sigma}_{x}$ ), i.e., the wave function $\varphi_{a, 1} \equiv \varphi_{a}$ can satisfy the periodic boundary condition; $\varphi_{a}^{\mathrm{t}}(x=\ell, t)=-\varphi_{a}^{\mathrm{t}}(x=0, t)$ and $\varphi_{a}^{\mathrm{b}}(x=\ell, t)=-\varphi_{a}^{\mathrm{b}}(x=0, t)\left(\hat{A}_{1}=-\hat{\sigma}_{x}\right)$, i.e., the wave function $\varphi_{a, 1} \equiv \varphi_{a}$ can also satisfy the antiperiodic boundary condition.

As was noted in section III, the (time-dependent) Weyl equations with the (self-adjoint) Hamiltonian operators $\hat{\mathrm{H}}_{a, 1}$ and $\hat{\mathrm{H}}_{a, 3}$ can provide purely real-valued solutions. Thus, if we impose on the respective wave functions the reality condition, these wave functions and their respective complex conjugates must satisfy the same boundary conditions, in which case the unitary matrices $\hat{A}_{1}$ and $\hat{A}_{3}$ must each be orthogonal. For example, in this case, the unitary matrix $\hat{A}_{1}$ in Eq. (C1) takes the form

$$
\hat{A}_{1}=\left[\begin{array}{cc}
m_{0} & -m_{2}  \tag{C3}\\
m_{2} & m_{0}
\end{array}\right]
$$

where $\left(m_{0}\right)^{2}+\left(m_{2}\right)^{2}=1$, and therefore, $\operatorname{det}\left(\hat{A}_{1}\right)=+1$ (because $m_{1}=m_{3}=0$ and $\mu=0$ ). Likewise, $\hat{A}_{1}$ in Eq. (C1) can also take the form

$$
\hat{A}_{1}=\left[\begin{array}{cc}
m_{3} & m_{1}  \tag{C4}\\
m_{1} & -m_{3}
\end{array}\right]
$$

where $\left(m_{1}\right)^{2}+\left(m_{3}\right)^{2}=1$, and therefore, $\operatorname{det}\left(\hat{A}_{1}\right)=-1$ (because $m_{0}=m_{2}=0$ and $\mu=\pi / 2$ ) [28]. Contrarily, the (time-dependent) Weyl equation with the (self-adjoint) Hamiltonian operator $\hat{H}_{a, 2}$ cannot provide real-valued solutions; thus, the corresponding wave functions support any boundary condition included in Eq. (33).

It is a known fact that families of boundary conditions for relativistic (and non-relativistic) Hamiltonian operators that describe a quantum particle inside a finite interval, and on the entire real line except the finite interval, are similar (in the latter case, if the interval is very small, we have the real line with a single point excluded - a hole - , and a particle in this kind of system can be modeled through proper boundary conditions only, and with potentials only - with singular potentials and with smooth potentials -). For example, in Ref. [32], 1D Dirac point interactions were recently modeled with a two-parameter potential that has a scalar part and an electrostatic part, and each is essentially a Dirac delta function. Likewise, in Ref. [33], some self-adjoint extensions of the 1D Dirac Hamiltonian operator
for massive "1D-fermions" confined to the interval $\Omega=[-L / 2,+L / 2]$ were recently studied (the latter results can be conveniently extended to the region $\mathbb{R}-\Omega$, which the authors call the dual geometry of $\Omega$ ). Essentially, the results in Ref. [33] were applied to the calculation of the so-called Casimir energy of a massive Dirac field confined in a 1D finite filament (of length $L$ ). Precisely, the most general family of (self-adjoint) boundary conditions for the Dirac Hamiltonian operator (in Eq. (3) of Ref. [33]) is given by

$$
\left[\begin{array}{l}
\psi_{1}(x=-L / 2)  \tag{C5}\\
\psi_{2}(x=+L / 2)
\end{array}\right]=\hat{\mathbb{U}} \hat{\gamma}_{0}\left[\begin{array}{l}
\psi_{1}(x=+L / 2) \\
\psi_{2}(x=-L / 2)
\end{array}\right]
$$

(see Eq. (23) in Ref. [33]), where $\hat{\mathbb{U}} \hat{\gamma}_{0}$ is a unitary matrix (this is because $\hat{\mathbb{U}}$ and $\hat{\gamma}_{0}$ are also unitary matrices). The latter family of boundary conditions is similar to the most general family of boundary conditions for the self-adjoint Weyl operator $\hat{\mathrm{H}}_{a, 3}$. In fact, the set of boundary conditions for $\hat{\mathrm{H}}_{a, 3}$ is given in Eq. (35), but it can also be written as the set in Eq. (C5), namely,

$$
\left[\begin{array}{l}
\varphi_{a}^{\mathrm{t}}(z=0)  \tag{C6}\\
\varphi_{a}^{\mathrm{b}}(z=\ell)
\end{array}\right]=\left(\hat{A}_{3} \hat{\sigma}_{x}\right)^{-1}\left[\begin{array}{l}
\varphi_{a}^{\mathrm{t}}(z=\ell) \\
\varphi_{a}^{\mathrm{b}}(z=0)
\end{array}\right],
$$

where $\left(\hat{A}_{3} \hat{\sigma}_{x}\right)^{-1}$ is a unitary matrix (this is because $\hat{A}_{3}$ and $\hat{\sigma}_{x}$ are also unitary matrices). For simplicity, the variable $t$ was dropped from the functions $\varphi_{a}^{\mathrm{t}}$ and $\varphi_{a}^{\mathrm{b}}$ in Eq. (C6). Again, all the boundary conditions for the (massive) 1D Dirac particle in the box are also valid for the (massless) 3D Weyl particle in the box.

Finally, as might be expected, within the family given in Eq. (C5), and in Eq. (C6), we also have the boundary condition commonly used in the so-called MIT bag model for hadronic structure (certainly, in its one-dimensional version) [34]. This confining boundary condition can be obtained from Eq. (C5), where the matrix $\hat{\mathbb{U}} \hat{\gamma}_{0}$ satisfies the relation $\hat{\mathbb{U}} \hat{\gamma}_{0} \hat{\alpha}+$ $\hat{\alpha} \hat{\mathbb{U}} \hat{\gamma}_{0}=\hat{0}_{2}$ (see Eq. (27) in Ref. [33]), by setting $\theta=-\pi / 2$ and $\eta=0$, and therefore, $\hat{\mathbb{U}} \hat{\gamma}_{0}=(-1)^{r}(-\mathrm{i}) \hat{\sigma}_{x}$. Thus, one obtains

$$
\begin{equation*}
\psi_{1}(-L / 2)=(-1)^{r}(-\mathrm{i}) \psi_{2}(-L / 2), \quad \psi_{1}(+L / 2)=(-1)^{r}(+\mathrm{i}) \psi_{2}(+L / 2) \tag{C7}
\end{equation*}
$$

Interestingly, the latter boundary condition can also be written explicitly in a form independent of the particular choice of the gamma matrices (and, as is known, also in a Lorentz-invariant way), namely,

$$
\begin{equation*}
\mathrm{i} n_{\mu} \hat{\gamma}^{\mu} \psi=\psi \text { at } x=-L / 2 \text { and } x=+L / 2 \tag{C8}
\end{equation*}
$$

where $n^{\mu}=(0,-1)$ at $x=-L / 2$, and $n^{\mu}=(0,+1)$ at $x=+L / 2$ (i.e., the unit twovector normal to the surface of the box is pointing outward from the wall). In addition, we have that $\hat{\gamma}^{0}=(-1)^{r} \hat{\sigma}_{x}$ and $\hat{\gamma}^{1}=\hat{\gamma}^{0} \hat{\alpha}=\hat{\gamma}^{0} \hat{\sigma}_{z}=(-1)^{r}(-\mathrm{i}) \hat{\sigma}_{y}$ (see Eq. (26) in Ref. [33]). Similarly, the usual MIT bag boundary condition can be obtained from the family in Eq. (C6) by setting $\left(\hat{A}_{3} \hat{\sigma}_{x}\right)^{-1}=(-\mathrm{i}) \hat{\sigma}_{x}$. Thus, one obtains $\varphi_{a}^{\mathrm{t}}(z=0)=(-\mathrm{i}) \varphi_{a}^{\mathrm{b}}(z=0)$, $\varphi_{a}^{\mathrm{t}}(z=\ell)=(+\mathrm{i}) \varphi_{a}^{\mathrm{b}}(z=\ell)$.

On the other hand, in $(1+1)$ dimensions, the most general family of self-adjoint boundary conditions for each of the (time-dependent) Weyl equations given in Eq. (36) is characterized by a phase, i.e., by a single real parameter. All boundary conditions present in these two families of boundary conditions cancel the boundary term in Eq. (39), which implies that

$$
\begin{equation*}
\left.\left.\left[\varphi_{a}^{*} \varphi_{a}\right]\right|_{0} ^{\ell} \equiv\left[\varrho_{a}\right]\right|_{0} ^{\ell}=0 \quad \Rightarrow \quad \varrho_{a}(x=\ell, t)=\varrho_{a}(x=0, t), \tag{C9}
\end{equation*}
$$

where $\varrho_{a}=\varrho_{a}(x, t)$ is the probability density. In this case, each Weyl equation leads to an atypical continuity equation, in which the probability density is precisely proportional to the probability current density, namely, $\partial\left(\varphi_{a}^{*} \varphi_{a}\right) / \partial t+(-1)^{a-1} \partial\left(c \varphi_{a}^{*} \varphi_{a}\right) / \partial x=0$. With that said, it is clear that all boundary conditions within the two one-parametric families of boundary conditions are non-confining boundary conditions, i.e., none of them can cancel the probability current density at the ends of the box.

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[1] H. Weyl, Gruppentheorie und Quantenmechanik, 1st Ed. (S. Hirzel, Leipzig, 1928). [in German]
[2] H. Weyl, "Gravitation and the electron", Proceedings of the National Academy of Sciences of the United States of America, 15, 323-34 (1929).
[3] H. Weyl, "Elektron und gravitation", Zeitschrift für Physik, 56, 330-52 (1929). [in German]
[4] I. E. Ovcharenko and Y. P. Stepanovsky, "On some properties of 2-D Weyl equation for charged massless spin $1 / 2$ particle", Problems of Atomic Science and Technology, N3, 56-60 (2007).
[5] S. De Bianchi, "Rethinking antiparticles. Hermann Weyl's contribution to neutrino physics", Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 61, 68-79 (2018).
[6] L. O'Raifeartaigh, The Dawning Of Gauge Theory (Princeton University Press, Princeton, 1997)
[7] T. D. Lee and C. N. Yang, "Parity Nonconservation and a two-component theory of the neutrino", Phys. Rev. 105, 1671-5 (1957).
[8] R. P. Feynman and M. Gell-Mann, "Theory of Fermi Interaction", Phys. Rev. 109, 193-8 (1958).
[9] J. J. Sakuray, Advanced Quantum Mechanics (Addison-Wesley, Reading, 1967).
[10] W. Greiner, Relativistic Quantum Mechanics: Wave Equations, 3rd Ed. (Springer, Berlin, 2000).
[11] E. Corinaldesi, Relativistic Wave Mechanics Edited by F. Strocchi. (Dover, New York, 2015).
[12] J. L. Díaz Cruz, et al., "Espinores de Weyl y el formalismo de helicidad", Revista Mexicana de Física E 61, 104-12 (2015). [in Spanish]
[13] L. H. Ryder, Quantum Field Theory, 2nd Ed. (Cambridge University Press, Cambridge, 1996).
[14] S. De Vincenzo, "On wave equations for the Majorana particle in $(3+1)$ and $(1+1)$ dimensions", arXiv:quant-ph/2007.03789v1 (2020). [Submitted for publication.]
[15] C. Giunti and C. W. Kim, Fundamentals of Neutrino Physics and Astrophysics (Oxford University Press, New York, 2007).
[16] P. B. Pal, "Dirac, Majorana, and Weyl fermions", Am. J. Phys. 79, 485-98 (2011).
[17] H. B. Nielsen and S. E. Rugh, "How do we live in 3+1 dimensions?", arXiv:hep-th/9407011v1 (1994).
[18] D. B. Kaplan, "Chiral symmetry and lattice fermions", arXiv:hep-lat/0912.2560v2 (2012).
[19] R. Jackiw and C. Rebbi, "Solitons with fermion number 1/2", Phys. Rev. D 13, 3398-3409 (1976).
[20] R. S. Bhalerao and B. Ram, "Fun and frustration with quarkonium in a $1+1$ dimension", Am. J. Phys. 69, 817-8 (2001).
[21] A. Smirnov and A. J. Dantas Farias Jr., "Representações da equação de Dirac em $1+1$ dimensões", Revista Brasileira de Ensino de Física 38, e3312 (2016). [in Portuguese]
[22] V. Alonso and S. De Vincenzo, "General boundary conditions for a Dirac particle in a box and
their non-relativistic limits", J. Phys. A: Math. Gen. 30, 8573-85 (1997).
[23] S. De Vincenzo, "Changes of representation and general boundary conditions for Dirac operators in $1+1$ dimensions", Revista Mexicana de Física 60, 401-8 (2014).
[24] G. Bonneau, J. Faraut and G. Valent, "Self-adjoint extensions of operators and the teaching of quantum mechanics", Am. J. Phys. 69, 322-31 (2001).
[25] J. Cayssol, "Introduction to Dirac materials and topological insulators", Comptes Rendus Physique 14, 760-78 (2013).
[26] R. D. Y. Hills et al. "From graphene and topological insulators to Weyl semimetals", in V. Dugaev, A. Wal and J. Barnas (eds.) Symmetry, Spin Dynamics and the Properties of Nanostructures: Lectures Notes of the 11th International School on Theoretical Physics (World Scientific Publishing, Singapore, 2015) pp. 277-315
[27] B. Thaller, Advanced Visual Quantum Mechanics (Springer, New York, 2005).
[28] S. De Vincenzo and C. Sánchez, "General boundary conditions for a Majorana single-particle in a box in $(1+1)$ dimensions", Physics of Particles and Nuclei Letters 15, 257-68 (2018).
[29] M. Asorey, D. García-Alvarez and J. M. Muñoz-Castañeda, "Boundary effects in bosonic and fermionic field theories", Int. J. Geom. Methods Mod. Phys. 12, 1560004 (2015).
[30] M. Asorey, A. Ibort and G. Marmo, "The topology and geometry of self-adjoint and elliptic boundary conditions for Dirac and Laplace operators", Int. J. Geom. Methods Mod. Phys. 12, 1561007 (2015).
[31] V. Alonso, S. De Vincenzo and L. Mondino, "On the boundary conditions for the Dirac equation", Eur. J. Phys. 18, 315-20 (1997).
[32] J. M. Guilarte, J. M. Muñoz-Castañeda, I. Pirozhenko and L. Santamaría-Sanz, "Onedimensional scattering of fermions on $\delta$-impurities", Front. Phys. 7, 109 (2019).
[33] M. Donaire, J. M. Muñoz-Castañeda, L. M. Nieto and M. Tello-Fraile, "Field fluctuations and Casimir energy of 1D-fermions", Symmetry 11, 643 (2019).
[34] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, "New extended model of hadrons", Phys. Rev. D 9, 3471-95 (1974).


[^0]:    *URL: https://orcid.org/0000-0002-5009-053X Electronic address: [salvatore.devincenzo@ucv.ve] ${ }^{\dagger}$ I would like to dedicate this paper to the memory of my beloved father Carmine De Vincenzo Di Fresca, who passed away unexpectedly on March 16, 2018. That day something inside of me also died.

