General boundary conditions for a Majorana single-particle in a box in (1+1) dimensions

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We consider the problem of a Majorana single-particle in a box in (1+1) dimensions. We show that the most general set of boundary conditions for the equation that models this particle is composed of two families of boundary conditions, each one with a real parameter. Within this set, we only have four confining boundary conditions - but infinite not confining boundary conditions. Our results are also valid when we include a Lorentz scalar potential in this equation. No other Lorentz potential can be added. We also show that the four confining boundary conditions for the Majorana particle are precisely the four boundary conditions that mathematically can arise from the general linear boundary condition used in the MIT bag model. Certainly, the four boundary conditions for the Majorana particle are also subject to the Majorana condition.

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I. INTRODUCTION

A Dirac single-particle that always moves inside a one-dimensional finite region, i.e., a Dirac particle in a one-dimensional box, is not a common problem in relativistic quantum mechanics. This is the case even though we do not need any external potential to confine the particle in this situation, only boundary conditions. Naturally, despite everything, this problem has been treated in various contexts; see [1–6]. An important result is that the most general family of boundary conditions for this type of particle (even when a bounded potential is present inside the box) has four (real) parameters; i.e., these boundary conditions are parametrized by a unitary matrix [5].

A Majorana particle is a massive fermion that is its own antiparticle; hence, it must be electrically neutral [7, 8]. Among the known spin-1/2 particles, only neutrinos could be of a Majorana nature [9]. However, Majorana fermions have recently emerged within condensed matter systems as exotic quasiparticle excitations; for instance, see [9–11] and the references therein. The problem of a Majorana single-particle in a one-dimensional box was also recently considered [12]. The authors of this article obtain the result that there are only four boundary conditions for Majorana fermions that are truly confined inside the box; i.e., there are only four types of walls that confine a Majorana fermion. However, there also exist boundary conditions that are not typical of a confined Majorana fermion, and these were not considered in Ref. [12]. In the present paper, we obtain the most general set of boundary conditions for the Majorana single-particle in a box.

The paper has been organized as follows. We begin the present section by introducing the Dirac equations (generally) physically associated to the Dirac particle and its antiparticle. Thus, here, we also introduce the so-called charge-conjugate wave function and the charge-conjugation matrix. Notice, however, that in a single-particle (electron) theory, the charge-conjugation operation changes a positive-energy electron state into a negative-energy electron state. Also, we obtain a formula that relates the matrices of charge-conjugation in any two representations with the respective unitary similarity matrix that changes the gamma matrices between these two representations. We particularize this formula by choosing an arbitrary representation as the representation of departure and the Majorana representation as the representation of arrival.

In section II, first, we introduce the Majorana condition, i.e., the condition that defines a Majorana particle or fermion. Then, we investigate the implications of imposing the Majorana condition upon the Dirac wave function in

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the Majorana and Dirac representations. On the one hand, we determine that in the Majorana representation, the Majorana condition implies that the Dirac wave function is real, and it satisfies a real Dirac equation. The latter can be considered as an equation for the Majorana particle. Additionally, we obtain that in the Dirac representation, the Majorana condition implies that the two components of the Dirac wave function are related, and hence, we obtain a complex equation of first-order in time and space for each of these components. One or the other can also be considered as an equation for the Majorana particle.

In section III, first, we note that any family of boundary conditions for the self-adjoint Dirac Hamiltonian operator in a box in the Dirac representation has a similar family of boundary conditions for the self-adjoint Dirac Hamiltonian operator in a box in the Majorana representation, and vice versa. Thus, both families of boundary conditions can be written in the same form. Because the most general family of boundary conditions in the Dirac representation is known [5], we can write the most general family of boundary conditions in the Majorana representation. Second, on the latter four-parameter family of boundary conditions, we impose the Majorana condition, and this leads us to the most general set of boundary conditions for the Majorana single-particle in a box (that is written in the Majorana basis). The latter is composed of two one-parameter families of boundary conditions. Within these families, we only have four (i.e., two plus two) confining boundary conditions, i.e., four boundary conditions that lead to the vanishing of the probability current density at the ends of the box (which were obtained for the first time in Ref. [12]), and many not confining boundary conditions. We also show that the four confining boundary conditions for the Majorana particle are precisely the four boundary conditions that mathematically can arise from the general linear boundary condition used in the MIT bag model [13–16]. Certainly, the four boundary conditions for the Majorana particle are also subject to the Majorana condition. Finally, we write the most general set of boundary conditions for the equation that models the Majorana particle (that is written in the Dirac basis). This set is composed of two one-parameter families of boundary conditions for a single function, i.e., for the solution to the equation that models the Majorana particle (for example, it could be the upper component of the Dirac wave function in the Dirac representation).

In section IV, we prove that the latter results are also valid if we add a Lorentz scalar potential to the equation for the Majorana particle, in fact, this latter potential is the only Lorentz potential that can be added to this equation. Finally, conclusions are presented in section V.

To begin, let ψ be the Dirac complex wave function of two components that satisfies the usual single-particle free Dirac equation in a covariant form

$$\left(\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}-\frac{\mathrm{m}c}{\hbar}\right)\psi=0. \tag{1}$$

Here, the matrices $\hat{\gamma}^{\mu}$, with $\mu=0,1$, are the Dirac gamma matrices [in (1+1) dimensions]. Specifically, $\hat{\gamma}^{0}\equiv\hat{\beta}$, $\hat{\gamma}^{1}\equiv\hat{\beta}\hat{\alpha}$, where the Dirac matrices, $\hat{\alpha}$ and $\hat{\beta}$, are Hermitian, and they satisfy relations $\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}=0$ and $\hat{\alpha}^{2}=\hat{\beta}^{2}=\hat{1}_{2}$ ($\hat{1}_{2}$ is the 2×2 identity matrix). Therefore, $\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}+\hat{\gamma}^{\nu}\hat{\gamma}^{\mu}=2g^{\mu\nu}\hat{1}_{2}$, where $g^{\mu\nu}=\mathrm{diag}(1,-1)$. Additionally, $(\hat{\gamma}^{\mu})^{\dagger}=\hat{\gamma}^{0}\hat{\gamma}^{\mu}\hat{\gamma}^{0}$ (symbol $\dagger$ indicates the Hermitian conjugate of a matrix and an operator). These last two relations imply that the gamma matrices are unitary, that is, $(\hat{\gamma}^{0})^{-1}=(\hat{\gamma}^{0})^{\dagger}$, and $(\hat{\gamma}^{1})^{-1}=(\hat{\gamma}^{1})^{\dagger}$ (but only $\hat{\gamma}^{0}$ is Hermitian, or self-adjoint, because it is a matrix, and $\hat{\gamma}^{1}$ is anti-Hermitian). The free Dirac equation in the Hamilton form,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi=\hat{\mathrm{h}}\psi, \tag{2}$$

with the corresponding Hamiltonian operator

$$\hat{\mathrm{h}}=-\mathrm{i}\hbar c\,\hat{\alpha}\,\frac{\partial}{\partial x}+\mathrm{m}c^{2}\hat{\beta}, \tag{3}$$

is obtained by multiplying Eq. (1) by the matrix $\hbar c\hat{\gamma}^{0}(\equiv\hbar c\hat{\beta})$ from the left and using the relations $(\hat{\gamma}^{0})^{2}=\hat{1}_{2}$ and $\hat{\gamma}^{0}\hat{\gamma}^{1}=\hat{\alpha}$.

The so-called charge-conjugate wave function [17, 18],

$$\psi_{C}\equiv\hat{S}_{C}\,\psi^{*}, \tag{4}$$

also satisfies Eq. (1),

$$\left(\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}-\frac{\mathrm{m}c}{\hbar}\right)\psi_{C}=0. \tag{5}$$

In the writing of Eq. (4), the raised asterisk * is used to denote the complex conjugate, and $\hat{S}_{C}$ is the charge-conjugation matrix. Because Eqs. (1) and (5) are equivalent (and equal in the free case), the matrices $\hat{\gamma}^{\mu}$ must satisfy the relation

$$\hat{S}_{C}\,(-\hat{\gamma}^{\mu})^{*}(\hat{S}_{C})^{-1}=\hat{\gamma}^{\mu}. \tag{6}$$

The matrices $\hat{\gamma}^\mu$ are unitary because $\hat{\gamma}^0\hat{\gamma}^1 + \hat{\gamma}^1\hat{\gamma}^0 = 0$ and $(\hat{\gamma}^0)^2 = -(\hat{\gamma}^1)^2 = \hat{1}_2$ and because $(\hat{\gamma}^0)^\dagger = \hat{\gamma}^0$ and $(\hat{\gamma}^1)^\dagger = -\hat{\gamma}^1$. Similarly, the matrices $(-\hat{\gamma}^\mu)^*$ are also unitary. In fact, because the metric $g^{\mu\nu}$ is real, we have $(-\hat{\gamma}^\mu)^*(-\hat{\gamma}^\nu)^* + (-\hat{\gamma}^\nu)^*(-\hat{\gamma}^\mu)^* = 2g^{\mu\nu}\hat{1}_2$; moreover, $((-\hat{\gamma}^\mu)^*)^\dagger = (-\hat{\gamma}^0)^*(-\hat{\gamma}^\mu)^*(-\hat{\gamma}^0)^*$. Therefore, $((-\hat{\gamma}^0)^*)^\dagger = ((-\hat{\gamma}^0)^*)^{-1}$ and $((-\hat{\gamma}^1)^*)^\dagger = ((-\hat{\gamma}^1)^*)^{-1}$. Because the matrices $\hat{\gamma}^\mu$ and $(-\hat{\gamma}^\mu)^*$ have to satisfy the relation (6), the matrix $\hat{S}_C$ can be chosen to be unitary [and it is defined up to a phase factor that belongs to the group $U(1)$ that is not fixed by the relation (6) itself]. Here, we will consider that $\hat{S}_C$ is unitary.

To pass one Dirac wave function that is written in a representation (or a basis), such as ψ, to another representation, such as ψ', we use the relation

$$\psi' = \hat{S}\,\psi. \tag{7}$$

Likewise, the matrices $\hat{\gamma}^\mu$ transform in the following way:

$$\hat{\gamma}^{\mu\,\prime} = \hat{S}\,\hat{\gamma}^\mu\,\hat{S}^{-1}. \tag{8}$$

The similarity matrix $\hat{S}$ can be chosen to be unitary because each $\hat{\gamma}^\mu$ is unitary (and it is defined up to a phase factor). The charge-conjugate wave function ψ_C can also be passed from one representation to another using the same relation (7), i.e.,

$$\psi'_C = \hat{S}\,\psi_C. \tag{9}$$

Thus, because we assumed that the relation that connects ψ with ψ_C has the form given by the Eq. (4),

$$\psi_C \equiv \hat{S}_C\,\psi^*, \tag{10}$$

we can also assume that the relation that connects ψ' with ψ'_C has the form

$$\psi'_C \equiv \hat{S}'_C\,(\psi')^*, \tag{11}$$

in which case, we have

$$\hat{S}'_C = \hat{S}\,\hat{S}_C\,(\hat{S}^*)^{-1}. \tag{12}$$

In fact, by substituting Eqs. (10) and (11) into Eq. (9), we obtain the relation $\hat{S}'_C(\psi')^* = \hat{S}\,\hat{S}_C\,\psi^*$, and by substituting in the latter the complex conjugate of Eq. (7), $(\psi')^* = \hat{S}^*\psi^*$, we finally obtain Eq. (12). Note that this formula is similar to the one that relates the gamma matrices in the two representations [Eq. (8)], but only when the matrix $\hat{S}$ is real.

The formula (12) permits us to pass the matrix $\hat{S}_C$ from one representation to another, provided the respective unitary similarity matrix that changes representation to the matrices $\hat{\gamma}^\mu$ is known ($\hat{S}$). Clearly, the matrix $\hat{S}_C$ (and also the matrix $\hat{S}$) is determined up to an arbitrary phase factor. Table 1 shows us, among other things, the results derived from the formula (12) by passing the matrix $\hat{S}_C$ from the Dirac representation (or standard) to the Majorana representation (and also to the Dirac representation itself). Additionally, matrix $\hat{S}$ in Table 1 refers to the matrix that permits us to pass from the Dirac representation to the Majorana representation (and also to the Dirac representation itself). In this paper, it will only be necessary to explicitly consider these two representations.

Representation	$\hat{\alpha}$	$\hat{\beta} \equiv \hat{\gamma}^0$	$\hat{\beta}\hat{\alpha} \equiv \hat{\gamma}^1$	$\hat{S}$	$\hat{S}_C$ (Dirac)	$\hat{S}'_C$
Dirac	$\hat{\sigma}_x$	$\hat{\sigma}_z$	$\mathrm{i}\hat{\sigma}_y$	$\hat{1}_2$	$\hat{\sigma}_x$	$\hat{\sigma}_x$
Majorana	$\hat{\sigma}_x$	$\hat{\sigma}_y$	$-\mathrm{i}\hat{\sigma}_z$	$\frac{1}{\sqrt{2}}(\hat{1}_2 + \mathrm{i}\hat{\sigma}_x)$	$-\mathrm{i}\hat{\sigma}_x$	$\hat{1}_2$

Table 1

Once again, notice that matrix $\hat{S}_C$ must only satisfy the relation $\hat{S}_C\,(-\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1} = \hat{\gamma}^\mu$ [Eq. (6)]; therefore, that matrix is determined up to a phase factor. If we change the phase factor of the matrix $\hat{S}_C$ in the Dirac representation, the matrix $\hat{S}_C$ that is obtained from formula (12) (identified here as $\hat{S}'_C$) changes in a factor that is also a phase.

Thus, the charge-conjugate wave function in any representation is given by $\psi_C \equiv \hat{S}_C\,\psi^*$ [Eq. (10)], where $\hat{S}_C = \hat{S}^{-1}\hat{S}'_C\hat{S}^*$ [the latter comes from Eq. (12)]. Then, if $\hat{S}$ is the unitary matrix that takes us from that representation (any representation) to the Majorana representation, we have $\hat{S}'_C = \hat{1}_2$; therefore, we can write the following result:

$$\psi_C \equiv \hat{S}_C\,\psi^* = \hat{S}^\dagger\hat{S}^*\psi^*. \tag{13}$$

Notice that $\hat{S}_C = \hat{S}^\dagger \hat{S}^*$ also verifies the relation

$$(\hat{S}_C)^\dagger = (\hat{S}_C)^*. \tag{14}$$

Furthermore, because $\hat{S}_C$ is a unitary matrix, the latter relation implies that $(\hat{S}_C)^{-1} = (\hat{S}_C)^*$. Clearly, if $\psi_C \equiv \hat{S}_C\, \psi^*$, we can require that $(\psi_C)_C = \psi$. This relation implies Eq. (14) because $\hat{S}_C$ is unitary.

II. THE MAJORANA PARTICLE

A Majorana particle or fermion is its own antiparticle. Therefore, the condition that defines a particle of this type is given by

$$\psi_C = \psi \tag{15}$$

(i.e., ψ is invariant under charge conjugation) [17]. The latter is called the Majorana condition (this condition is sometimes written as $\psi_C = \omega\psi$, where ω is an arbitrary phase factor). In any representation, this condition implies the result

$$\psi = \hat{S}_C\, \psi^* = \hat{S}^\dagger \hat{S}^* \psi^*, \tag{16}$$

where we used Eq. (13), and the matrix $\hat{S}$ is the unitary matrix that takes us from any representation to the Majorana representation.

We can investigate the implications of imposing the Majorana condition upon the Dirac wave function in the Majorana and Dirac representations. For example, in the Majorana representation, we write the Majorana condition as follows

$$\psi'_C = \psi' \tag{17}$$

(in this case, we use primes to identify quantities in the Majorana representation). We write Eq. (16) as follows

$$\psi' = (\hat{S})^\dagger (\hat{S})^* (\psi')^*. \tag{18}$$

Because the unitary matrix that takes us from the Majorana representation to the same Majorana representation is $\hat{S} = \hat{1}_2$ ($\Rightarrow \hat{S}'_C = \hat{1}_2$), the following result is obtained:

$$\psi' = (\psi')^*, \tag{19}$$

i.e., the Majorana condition imposed upon the Dirac wave function in the Majorana representation implies that this wave function must be real. With this condition, we write the Dirac wave function ψ' in the form

$$\psi' \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} (\phi_1)^* \\ (\phi_2)^* \end{bmatrix}. \tag{20}$$

This wave function is the solution of the free Dirac equation (1) [or Eq. (2)] in the Majorana representation, with the Majorana condition, namely,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{i}\hbar c \frac{\partial}{\partial x} - \mathrm{i}mc^2 \\ -\mathrm{i}\hbar c \frac{\partial}{\partial x} + \mathrm{i}mc^2 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \hat{\mathrm{h}}' \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{21}$$

Clearly, in this case, the Dirac equation is a system of two coupled equations that is satisfied by the components ϕ_1 and ϕ_2, and also by the components $(\phi_1)^*$ and $(\phi_2)^*$; i.e., Eq. (21) is a real equation (the latter because of the Majorana condition). It is worth mentioning that in one Majorana basis, the gamma matrices must satisfy the relation $(\mathrm{i}\hat{\gamma}^\mu)^* = \mathrm{i}\hat{\gamma}^\mu$ ($\Rightarrow -(\hat{\gamma}^\mu)^* = \hat{\gamma}^\mu$). Of course, the representation here that is considered the Majorana representation satisfies that relation (see Table 1). The latter implies that the Dirac operator $\mathrm{i}\hat{\gamma}^\mu \partial_\mu - \frac{\mathrm{m}c}{\hbar}$ is real. Thus, just for choosing matrices in the Majorana representation, the solutions of Eq. (1) can be chosen to be real. Finally, Eq. (21), with its real-valued solutions, can also be considered as an equation for the Majorana particle. On a side note, the Majorana representation used in the present article differs slightly from that used in Ref. [12]. In fact, in the latter reference (and using the symbols used therein), $\hat{\tilde{\gamma}}^0 (= \hat{\sigma}_y)$ remains the same, but $\hat{\tilde{\gamma}}^1 (= +\mathrm{i}\hat{\sigma}_z)$ changes. Precisely, these

two representations verify $\hat{\tilde{\gamma}}^{\mu} = \hat{\sigma}_y \, \hat{\gamma}^{\mu\,\prime} \, \hat{\sigma}_y$, and $\left[\, \tilde{\psi}_1 \; \tilde{\psi}_2 \,\right]^{\mathrm{T}} = \hat{\sigma}_y \left[\, \phi_1 \; \phi_2 \,\right]^{\mathrm{T}}$ (the symbol T represents the transpose of a matrix).

In the Dirac representation, the Majorana condition, Eq. (15) or Eq. (16), leads us to the following relation:

$$\psi = \hat{S}^{\dagger} \hat{S}^{*} \psi^{*} = (\hat{S}^{*})^{2} \psi^{*} = -\mathrm{i}\hat{\sigma}_x \psi^{*}. \tag{22}$$

The latter, because of the unitary matrix that takes us from the Dirac representation to the Majorana representation is given by $\hat{S} = \frac{1}{\sqrt{2}}(\hat{1}_2 + \mathrm{i}\hat{\sigma}_x)$ ($\Rightarrow \hat{S}_C = -\mathrm{i}\hat{\sigma}_x$). That is, the Majorana condition imposed upon the Dirac wave function in the Dirac representation implies that

$$\psi \equiv \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = -\mathrm{i}\hat{\sigma}_x \begin{bmatrix} \varphi^{*} \\ \chi^{*} \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\chi^{*} \\ -\mathrm{i}\varphi^{*} \end{bmatrix}, \tag{23}$$

and this matrix condition imposes the following relation between the components of ψ:

$$\chi = -\mathrm{i}\varphi^{*} \, (\Leftrightarrow \, \varphi = -\mathrm{i}\chi^{*}). \tag{24}$$

Therefore, in this case, the two-component Dirac wave function could be written as

$$\psi = \begin{bmatrix} \varphi \\ -\mathrm{i}\varphi^{*} \end{bmatrix}, \tag{25}$$

that is, the solution of the free Dirac equation (1) [or Eq. (2)] in the Dirac representation,

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \begin{bmatrix} \varphi \\ -\mathrm{i}\varphi^{*} \end{bmatrix} = \begin{bmatrix} \mathrm{m}c^2 & -\mathrm{i}\hbar c \frac{\partial}{\partial x} \\ -\mathrm{i}\hbar c \frac{\partial}{\partial x} & -\mathrm{m}c^2 \end{bmatrix} \begin{bmatrix} \varphi \\ -\mathrm{i}\varphi^{*} \end{bmatrix} = \hat{\mathrm{h}} \begin{bmatrix} \varphi \\ -\mathrm{i}\varphi^{*} \end{bmatrix}. \tag{26}$$

Eq. (26) implies that φ satisfies the following equation:

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \varphi = -\hbar c \frac{\partial}{\partial x} \varphi^{*} + \mathrm{m}c^2 \varphi. \tag{27}$$

If we choose to write the wave function as $\psi = \left[\, -\mathrm{i}\chi^{*} \; \chi \,\right]^{\mathrm{T}}$, the equation for χ is equal to Eq. (27) but with the replacement of $\mathrm{m} \to -\mathrm{m}$. In any case, it is enough to solve at least one of these two first-order equations because φ and χ are algebraically related by the relation (24).

Equation (27) alone could be called the Majorana equation because it models the Majorana particle. However, this equation is not precisely the equation that is known in the literature as the Majorana equation [17]. The latter specifically reads

$$\mathrm{i}\hat{\gamma}^{\mu} \partial_{\mu} \psi - \frac{\mathrm{m}c}{\hbar} \psi_C = 0. \tag{28}$$

In any case, Eq. (27) is the equation for the Majorana particle in the Dirac representation. The Majorana condition in the Dirac basis, $\psi_C = \psi \Rightarrow -\mathrm{i}\hat{\sigma}_x \psi^{*} = \psi$, is what leads to the equation for a single component of ψ, i.e., to Eq. (27). Certainly, Eq. (28) together with the Majorana condition also leads us to the equation for the Majorana particle in any representation [for example, to Eq. (21) in the Majorana representation and to Eq. (27) in the Dirac representation].

III. THE MAJORANA (FREE) PARTICLE IN A BOX

The free Dirac Hamiltonian in (3), in the interval $x \in \Omega = [a, b]$ (a box), is an (unbounded) Hermitian, or formally a self-adjoint, differential operator. In fact, because the matrices $\hat{\alpha}$ and $\hat{\beta}$ (i.e., the bounded operators $\hat{\alpha}$ and $\hat{\beta}$) are Hermitian and because we formally have $\hat{\mathrm{p}} \equiv -\mathrm{i}\hbar\partial/\partial x = \hat{\mathrm{p}}^{\dagger}$ (i.e., without the specification of its domain), we also formally have $\hat{\mathrm{h}} = \hat{\mathrm{h}}^{\dagger}$. The Hamiltonian acts on two-component column vectors $\psi = \psi(x, t)$ that belong to the Hilbert space of the square integrable functions, $\mathcal{H} = \mathcal{L}^2(\Omega) \oplus \mathcal{L}^2(\Omega)$, with a scalar product denoted by $\langle \psi, \xi \rangle \equiv \int_{\Omega} \mathrm{d}x \, \psi^{\dagger} \xi$, and norm $\| \, \psi \, \| \equiv \sqrt{\langle \psi, \psi \rangle}$, and also, $\hat{\mathrm{h}}\psi \in \mathcal{H}$. Additionally, if $\hat{\mathrm{h}}$ is self-adjoint, ψ must only satisfy specific boundary conditions at the ends of the interval Ω. These conditions define the so-called domain of the Hamiltonian, $D(\hat{\mathrm{h}})$. In fact, $\hat{\mathrm{h}}$ is self-adjoint because $\hat{\mathrm{h}} = \hat{\mathrm{h}}^{\dagger}$, but then, we also have $D(\hat{\mathrm{h}}) = D(\hat{\mathrm{h}}^{\dagger})$ (for example, see Ref. [19]).

It can be demonstrated that $\hat{\mathrm{h}}$ satisfies the following relation:

$$\langle \psi, \hat{\mathrm{h}}\xi \rangle - \langle \hat{\mathrm{h}}\psi, \xi \rangle = -\mathrm{i}\hbar c \left[\psi^{\dagger} \hat{\alpha}\, \xi \right]\Big|_a^b, \tag{29}$$

where $[\, f \,]\big|_a^b \equiv f(b,t) - f(a,t)$. If the boundary conditions imposed upon ψ and ξ lead to the cancellation of the term that is evaluated at the ends of the interval Ω in (29), $\langle \psi, \hat{\mathrm{h}}\xi \rangle = \langle \hat{\mathrm{h}}\psi, \xi \rangle$; i.e., the operator $\hat{\mathrm{h}}$ is Hermitian [remember that $\langle \psi, \hat{\mathrm{h}}\xi \rangle = \langle \hat{\mathrm{h}}^{\dagger}\psi, \xi \rangle$ is, *grosso modo*, the relation that defines the adjoint operator $\hat{\mathrm{h}}^{\dagger}$ on a vector space]. If we make $\psi = \xi$ in the latter relation, we obtain $\langle \psi, \hat{\mathrm{h}}\psi \rangle = \langle \hat{\mathrm{h}}\psi, \psi \rangle = \langle \psi, \hat{\mathrm{h}}\psi \rangle^*$ [$\Rightarrow \langle \psi, \hat{\mathrm{h}}\psi \rangle \equiv \langle \hat{\mathrm{h}} \rangle_{\psi} \in \mathbb{R}$, which is an expected result for a Hermitian operator]. For the same reason, we have

$$c \left[\psi^{\dagger} \hat{\alpha}\, \psi \right]\Big|_a^b \equiv [\, j \,]\big|_a^b = 0 \quad \Rightarrow \quad j(b,t) = j(a,t), \tag{30}$$

where $j = j(x,t) \equiv c\psi^{\dagger}\hat{\alpha}\,\psi$ is the probability current density. Any state in the domain $D(\hat{\mathrm{h}})$ must satisfy the condition given in Eq. (30) that depends only on the matrix $\hat{\alpha}$, i.e., that condition does not depend on the matrix $\hat{\beta}$. Therefore, any family of boundary conditions for the self-adjoint Dirac Hamiltonian in a box in the Dirac representation ($\hat{\alpha} = \hat{\sigma}_x$),

$$\hat{\mathrm{h}} = -\mathrm{i}\hbar c\, \hat{\sigma}_x \frac{\partial}{\partial x} + \mathrm{m}c^2 \hat{\sigma}_z, \tag{31}$$

has a similar family of boundary conditions for the self-adjoint Dirac Hamiltonian in a box in the Majorana representation ($\hat{\alpha}' = \hat{\alpha} = \hat{\sigma}_x$),

$$\hat{\mathrm{h}}' = \hat{S}\, \hat{\mathrm{h}}\, \hat{S}^{\dagger} = -\mathrm{i}\hbar c\, \hat{\sigma}_x \frac{\partial}{\partial x} + \mathrm{m}c^2 \hat{\sigma}_y, \tag{32}$$

and vice versa (here, the word similar means that both families of boundary conditions can be written in the same form). We know that for the first Hamiltonian ($\hat{\mathrm{h}}$) the most general family of boundary conditions has the following format (we omit the variable t in any component of the Dirac wave function and in j hereinafter)

$$\begin{bmatrix} \varphi(b) + \chi(b) \\ \varphi(a) - \chi(a) \end{bmatrix} = \hat{U} \begin{bmatrix} \varphi(b) - \chi(b) \\ \varphi(a) + \chi(a) \end{bmatrix}, \tag{33}$$

where $\hat{U}$ is a matrix that belongs to the group $U(2)$ (see Refs. [5, 6]). Thus, for the second Hamiltonian ($\hat{\mathrm{h}}'$) that is written in the Majorana representation, the most general family of boundary conditions can have the following format (the same format as in the precedent family)

$$\begin{bmatrix} \phi_1(b) + \phi_2(b) \\ \phi_1(a) - \phi_2(a) \end{bmatrix} = \hat{U}' \begin{bmatrix} \phi_1(b) - \phi_2(b) \\ \phi_1(a) + \phi_2(a) \end{bmatrix}, \tag{34}$$

where $\hat{U}'$ is a unitary matrix and has four real parameters. To obtain Eq. (34), in Eq. (33) we made the replacements $\varphi \to \phi_1$, $\chi \to \phi_2$, and $\hat{U} \to \hat{U}'$. An achievable choice (or parametrization) for the matrix $\hat{U}'$ is given by

$$\hat{U}' = \exp(\mathrm{i}\mu) \begin{bmatrix} m_0 - \mathrm{i}m_3 & -m_2 - \mathrm{i}m_1 \\ m_2 - \mathrm{i}m_1 & m_0 + \mathrm{i}m_3 \end{bmatrix}, \tag{35}$$

where $\mu \in [0, \pi)$ and the other real quantities (m_0, m_1, m_2, and m_3) satisfy $(m_0)^2 + (m_1)^2 + (m_2)^2 + (m_3)^2 = 1$. Note that the unitary matrix $\hat{U}'$ verifies $\det(\hat{U}') = \exp(\mathrm{i}2\mu)$. The family of boundary conditions (33) or (34) is the most general family of boundary conditions for a Dirac particle in a box.

As we observed in section II, the Majorana condition in the Majorana basis, $\psi'_C = \psi'$, imposed upon the corresponding Dirac wave function, $\psi' = [\, \phi_1 \;\; \phi_2 \,]^{\mathrm{T}}$, implies that this wave function is real ($\psi' = (\psi')^*$). Thus, both ϕ_1 and ϕ_2 as $(\phi_1)^*$ and $(\phi_2)^*$ satisfy the general boundary condition (34), in which case the matrix $\hat{U}'$ must be real

$$\hat{U}' = (\hat{U}')^*. \tag{36}$$

The latter condition implies that the matrix $\hat{U}'$ must be orthogonal, i.e., $(\hat{U}')^{\dagger} = ((\hat{U}')^*)^{\mathrm{T}} = (\hat{U}')^{\mathrm{T}} = (\hat{U}')^{-1}$. Therefore, the determinant of $\hat{U}'$ is either $+1$ or -1. Particularly, in the case $\det(\hat{U}') = 1$ ($\mu = 0$), the matrix $\hat{U}'$ satisfying the reality condition takes the form

$$\hat{U}' = \begin{bmatrix} m_0 & -m_2 \\ m_2 & m_0 \end{bmatrix}, \tag{37}$$

where $(m_0)^2+(m_2)^2 = 1$, i.e., $m_1 = m_3 = 0$. Thus, $\hat{U}'$ belongs to the group $SO(2)$. Likewise, in the case $\det(\hat{U}') = -1$ ($\mu = \pi/2$) the matrix $\hat{U}'$ takes the form

$$\hat{U}' = \begin{bmatrix} m_3 & m_1 \\ m_1 & -m_3 \end{bmatrix}, \tag{38}$$

where $(m_1)^2 + (m_3)^2 = 1$, i.e., $m_0 = m_2 = 0$. Thus, the first general boundary condition for the Majorana particle or fermion in the Majorana basis can be written as:

$$\begin{bmatrix} \phi_1(b) + \phi_2(b) \\ \phi_1(a) - \phi_2(a) \end{bmatrix} = \begin{bmatrix} m_0 & -m_2 \\ m_2 & m_0 \end{bmatrix} \begin{bmatrix} \phi_1(b) - \phi_2(b) \\ \phi_1(a) + \phi_2(a) \end{bmatrix}, \tag{39}$$

where ϕ_1 and ϕ_2 are real functions. Clearly, this is a one-parameter family of boundary conditions. Note that if we had made the choice $\mu \in [0, \pi]$, we would have obtained the term (± 1) in front of the square matrix in Eq. (38). The sign $(+)$ would correspond to the case $\mu = 0$; and the sign $(-)$ to the case $\mu = \pi$. These two cases give rise to two identical one-parameter families of boundary conditions. Likewise, the second general boundary condition for the Majorana particle in the Majorana basis can also be written as:

$$\begin{bmatrix} \phi_1(b) + \phi_2(b) \\ \phi_1(a) - \phi_2(a) \end{bmatrix} = \begin{bmatrix} m_3 & m_1 \\ m_1 & -m_3 \end{bmatrix} \begin{bmatrix} \phi_1(b) - \phi_2(b) \\ \phi_1(a) + \phi_2(a) \end{bmatrix}, \tag{40}$$

where ϕ_1 and ϕ_2 are real functions. The latter is also a one-parameter family of boundary conditions. These two general families of boundary conditions describe all the possible boundary conditions for the Majorana particle in the Majorana basis.

Within Eqs. (39) and (40), there exist confining and not confining boundary conditions. On the one hand, the called confining conditions lead to the vanishing of the probability current density at the ends of the box, i.e.,

$$j(b) = j(a) = 0. \tag{41}$$

It is easy to see that $j = c\psi^\dagger \hat{\alpha}\, \psi$ is invariant under a change of unitary representation [6]. Thus, the impenetrability condition in (41) does not change by changes of representation. On the other hand, we already know that for the boundary conditions that verify the condition in (41), the matrix $\hat{U}$ in (33) is a diagonal matrix (see Ref. [5]). Then, the matrix $\hat{U}'$ in (34) must also be diagonal for that condition to be satisfied. Incidentally, by explicitly imposing the latter condition of diagonality on the matrix $\hat{U}$ in (33), it can be shown that a two-parameter subfamily of confining boundary conditions is obtained, not a one-parameter subfamily. Thus, if the matrix $\hat{U}'$ in (34), i.e., the matrix (37), is diagonal, we have $m_2 = 0$, and therefore, $m_0 = \pm 1$. Consequently,

$$\hat{U}' = \pm \hat{1}_2 \quad (\mu = 0). \tag{42}$$

These results give us the following two boundary conditions:

$$\phi_2(a) = \phi_2(b) = 0, \tag{43}$$

in the case of $m_0 = 1$. In addition,

$$\phi_1(a) = \phi_1(b) = 0, \tag{44}$$

in the case of $m_0 = -1$. Likewise, if the matrix $\hat{U}'$ in (34), i.e., the matrix (38), is diagonal, we have $m_1 = 0$, and therefore, $m_3 = \pm 1$. Consequently,

$$\hat{U}' = \pm \hat{\sigma}_z \quad (\mu = \pi/2). \tag{45}$$

These results give us the following two boundary conditions:

$$\phi_1(a) = \phi_2(b) = 0, \tag{46}$$

in the case of $m_3 = 1$. In addition,

$$\phi_2(a) = \phi_1(b) = 0, \tag{47}$$

in the case of $m_3 = -1$.

These two pairs of boundary conditions can be passed to the Dirac basis via the matricial relation

$$\psi' = \frac{1}{\sqrt{2}}(\hat{1}_2 + \mathrm{i}\hat{\sigma}_x)\psi \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \begin{bmatrix} \varphi \\ \chi \end{bmatrix}. \tag{48}$$

Therefore, the boundary condition in (43) takes the form

$$\varphi(a) = +\mathrm{i}\chi(a)\,, \quad \varphi(b) = +\mathrm{i}\chi(b), \tag{49}$$

and the condition in (44) is

$$\varphi(a) = -\mathrm{i}\chi(a)\,, \quad \varphi(b) = -\mathrm{i}\chi(b). \tag{50}$$

Likewise, the boundary condition in (46) takes the form

$$\varphi(a) = -\mathrm{i}\chi(a)\,, \quad \varphi(b) = +\mathrm{i}\chi(b), \tag{51}$$

and the condition in (47) is

$$\varphi(a) = +\mathrm{i}\chi(a)\,, \quad \varphi(b) = -\mathrm{i}\chi(b). \tag{52}$$

From the Majorana condition in the Majorana representation, $\psi' = (\psi')^*$ [Eq. (19)], and using Eq. (48), we obtain $\chi = -\mathrm{i}\varphi^* (\Leftrightarrow \varphi = -\mathrm{i}\chi^*)$. Thus, the functions φ and χ in Eqs. (49)-(52) must obey the latter relation (which obviously is the Majorana condition in the Dirac representation [Eq. (24)]). Using this relation, we can write the boundary conditions in equations (49)-(52), for example, only in terms of φ,

$$\varphi(a) = \varphi^*(a)\,, \quad \varphi(b) = \varphi^*(b), \tag{53}$$

which comes from the condition in (49), and analogously,

$$\varphi(a) = -\varphi^*(a)\,, \quad \varphi(b) = -\varphi^*(b), \tag{54}$$

which comes from the condition in (50). Likewise,

$$\varphi(a) = -\varphi^*(a)\,, \quad \varphi(b) = \varphi^*(b), \tag{55}$$

which comes from the condition in (51), and analogously,

$$\varphi(a) = \varphi^*(a)\,, \quad \varphi(b) = -\varphi^*(b), \tag{56}$$

which comes from the condition in (52). The boundary conditions (53)-(56) are precisely the boundary conditions for the equation that describes the Majorana particle, i.e., for Eq. (27). The latter four boundary conditions were also recently obtained in Ref. [12].

It is very interesting to note that these four apparently uncommon boundary conditions are actually typical boundary conditions. Specifically, the boundary condition in (53) is a Dirichlet boundary condition imposed upon the function $\mathrm{Im}(\varphi)$, i.e.,

$$\mathrm{Im}(\varphi(a)) = \mathrm{Im}(\varphi(b)) = 0. \tag{57}$$

Likewise, the boundary condition in (54) is a Dirichlet boundary condition imposed upon the function $\mathrm{Re}(\varphi)$, i.e.,

$$\mathrm{Re}(\varphi(a)) = \mathrm{Re}(\varphi(b)) = 0. \tag{58}$$

On the other hand, the boundary conditions in (55) and (56) are mixed boundary conditions, i.e.,

$$\mathrm{Re}(\varphi(a)) = \mathrm{Im}(\varphi(b)) = 0, \tag{59}$$

and

$$\mathrm{Im}(\varphi(a)) = \mathrm{Re}(\varphi(b)) = 0, \tag{60}$$

respectively.

For the problem of a Dirac single-particle in a one-dimensional box, the confining boundary conditions are characterized by the condition given by Eq. (41), i.e., the probability current density must be zero at the ends of the box. The latter condition can also be written as

$$n_\mu \bar{\psi}\hat{\gamma}^\mu\psi = 0 \text{ at } x = a \text{ and } x = b, \tag{61}$$

where $\bar{\psi} \equiv \psi^\dagger\hat{\gamma}^0 = \psi^\dagger\hat{\beta}$ is the Dirac adjoint of ψ, and $n^\mu = (n^0, n^1) = (0, \pm 1)$ is a unit two-vector normal to the surface of the confining region. Indeed [in (1+1) dimensions],

$$n_\mu \bar{\psi}\hat{\gamma}^\mu\psi = n_0\bar{\psi}\hat{\gamma}^0\psi + n_1\bar{\psi}\hat{\gamma}^1\psi = 0 - n^1\bar{\psi}\hat{\gamma}^1\psi = \mp\bar{\psi}\hat{\gamma}^1\psi = \mp\psi^\dagger\hat{\gamma}^0\hat{\gamma}^1\psi = \mp\psi^\dagger\hat{\alpha}\psi = 0 \quad \Rightarrow \quad j = 0,$$

at $x = a$ and $x = b$.

In the so-called MIT bag model for hadronic structure [13–16] (although here we have it in its one-dimensional version [4, 5, 20]), the condition (61) is satisfied by imposing the following general linear boundary condition:

$$\mathrm{i}\, n_\mu\hat{\gamma}^\mu\psi = \psi \text{ at } x = a \text{ and } x = b, \tag{62}$$

nevertheless, this implies that $\bar{\psi}\psi = \psi^\dagger\hat{\beta}\psi$ is also zero at those points. Hence, the confining boundary conditions of the MIT bag model allow the Hamiltonian $\hat{\mathrm{h}}'$ in (32) (and $\hat{\mathrm{h}}$ in (31)), as well as the operator $c\hat{\beta}\hat{\mathrm{p}} = -\mathrm{i}\hbar c\hat{\beta}\,\partial/\partial x$, to be self-adjoint [Incidentally, the latter operator is real in the Majorana basis]. A complete discussion about one-dimensional self-adjoint Dirac operators and their boundary conditions can be found in Ref. [6].

Usually, we choose $n^\mu = (0, -1)$ at $x = a$, and $n^\mu = (0, 1)$ at $x = b$, i.e., the unit two-vector normal to the surface of the box is pointing outward from the wall. Thus, the boundary condition given by Eq. (62) takes the form

$$+\,\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = a, \text{ and } -\,\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = b \tag{63}$$

(with primes, i.e., $\psi \to \psi'$, etc., labeling quantities in the Majorana representation) [4, 5]. The latter is precisely the boundary condition commonly used in the MIT bag model. Precisely, in the Majorana representation ($\hat{\alpha}' = \hat{\sigma}_x$, $\hat{\beta}' = \hat{\sigma}_y$), this boundary condition takes the form $\phi_2(a) = 0$, $\phi_1(b) = 0$. To obtain the latter boundary condition from the most general family of boundary conditions in (34) (with the matrix $\hat{U}'$ in (35)), we must make $m_0 = m_1 = m_2 = 0$, $m_3 = -1$, and $\mu = \pi/2$ ($\Rightarrow \hat{U}' = -\hat{\sigma}_z$). In the Dirac representation this boundary condition is $\varphi(a) = +\mathrm{i}\chi(a)$, $\varphi(b) = -\mathrm{i}\chi(b)$. It is noteworthy that the boundary condition for the Dirac particle we have presented in this paragraph, with the imposition of the Majorana condition, is precisely one of the four confining boundary conditions found for the Majorana particle in the box [Eq. (47), or Eq. (52)]. In relation to this result, remember that ϕ_1 and ϕ_2 must be real-valued functions in the Majorana representation; this last condition arises from the imposition of the Majorana condition upon the Dirac wave function. On the other hand, the Majorana condition in the Dirac representation implies that φ and χ relate to each other [see Eq. (24)].

Obviously, we can also choose $n^\mu = (0, 1)$ at $x = a$, and $n^\mu = (0, -1)$ at $x = b$, i.e., in this case the unit two-vector normal to the surface of the box is pointing inward from the wall. Therefore,

$$-\,\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = a, \text{ and } +\,\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = b. \tag{64}$$

In the Majorana representation, the latter boundary condition takes the form $\phi_1(a) = 0$, $\phi_2(b) = 0$. This boundary condition can be obtained from Eqs. (34) and (35) by setting $m_0 = m_1 = m_2 = 0$, $m_3 = +1$, and $\mu = \pi/2$ ($\Rightarrow \hat{U}' = +\hat{\sigma}_z$). In the Dirac representation this boundary condition is $\varphi(a) = -\mathrm{i}\chi(a)$, $\varphi(b) = +\mathrm{i}\chi(b)$. The boundary condition for the Dirac particle we have presented in this paragraph, with the imposition of the Majorana condition, again is one of the four confining boundary conditions found for the Majorana particle in the box [Eq. (46), or Eq. (51)]. It was recently shown that the free Dirac Hamiltonian operator $\hat{\mathrm{h}}(\mathrm{m})$ with the boundary condition in (63) is unitarily equivalent to the free Dirac operator $\hat{\mathrm{h}}(-\mathrm{m})$ with the boundary condition in (64) [21]. Thus, the spectral behavior of the MIT bag model depends on the sign of m, or equivalently, the orientation of the normal.

From a strictly mathematical point of view, we could also choose the unit two-vector normal in the form $n^\mu = (0, -1)$ at $x = a$, and also at $x = b$. Thus, we obtain the following boundary condition:

$$+\,\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = a \text{ and } x = b. \tag{65}$$

In the Majorana representation, this boundary condition takes the form $\phi_2(a) = 0$, $\phi_2(b) = 0$. To obtain the latter boundary condition from the most general family of boundary conditions in (34) (with the matrix $\hat{U}'$ in (35)), we must make $m_0 = +1$, $m_1 = m_2 = m_3 = 0$, and $\mu = 0$ ($\Rightarrow \hat{U}' = +\hat{1}_2$). In the Dirac representation this boundary condition is $\varphi(a) = +\mathrm{i}\chi(a)$, $\varphi(b) = +\mathrm{i}\chi(b)$. The boundary condition for the Dirac particle we have presented in this paragraph, with the imposition of the Majorana condition, again is one of the four confining boundary conditions found for the Majorana particle in the box [Eq. (43), or Eq. (49)].

Likewise, we could also choose the unit two-vector normal in the form $n^\mu = (0, 1)$ at $x = a$, and also at $x = b$. In this case we obtain

$$-\mathrm{i}\hat{\beta}\hat{\alpha}\psi = \psi \text{ at } x = a \text{ and } x = b. \tag{66}$$

In the Majorana representation, the latter boundary condition has the form $\phi_1(a) = 0$, $\phi_1(b) = 0$. This boundary condition can be obtained from Eqs. (34) and (35) by setting $m_0 = -1$, $m_1 = m_2 = m_3 = 0$, and $\mu = 0$ ($\Rightarrow \hat{U}' = -\hat{1}_2$). In the Dirac representation this boundary condition is $\varphi(a) = -\mathrm{i}\chi(a)$, $\varphi(b) = -\mathrm{i}\chi(b)$. The boundary condition for the Dirac particle we have presented in this paragraph, with the imposition of the Majorana condition, again is one of the four confining boundary conditions found for the Majorana particle in the box [Eq. (44), or Eq. (50)].

We can also obtain not confining boundary conditions for the Majorana particle in a box. The latter do not explicitly lead to the vanishing of the probability current density at the ends of the box. In this situation, the particle certainly moves inside the box, but the ends could be seen to be physically connected; for example, the particle could hit one wall and reappear at the other. First, we rewrite the boundary conditions in (39) and (40) in the following manner:

$$\begin{bmatrix} \phi_1(b) \\ \phi_2(b) \end{bmatrix} = \frac{1}{m_2} \begin{bmatrix} 0 & -m_0 - 1 \\ m_0 - 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix}, \tag{67}$$

which comes from Eq. (39), with $(m_0)^2 + (m_2)^2 = 1$. Likewise,

$$\begin{bmatrix} \phi_1(b) \\ \phi_2(b) \end{bmatrix} = \frac{1}{m_1} \begin{bmatrix} m_3 + 1 & 0 \\ 0 & -m_3 + 1 \end{bmatrix} \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix}, \tag{68}$$

which comes from Eq. (40), with $(m_1)^2 + (m_3)^2 = 1$.

Notice that the 2×2 matrix that connects the column vectors present in (67) is equal to its own inverse; that is to say, the inverse expression of (67) is simply obtained making the replacements $a \to b$ and $b \to a$. On the other hand, the inverse matrix of the 2×2 matrix in (68) is obtained from the latter making the replacement $m_3 \to -m_3$. As we observed before, two confining boundary conditions emerged from Eq. (39) by making $m_2 = 0$, and two others from Eq. (40) by making $m_1 = 0$. Then, the boundary conditions (67) and (68) become not confining boundary conditions only when $m_2 \neq 0$ and $m_1 \neq 0$, respectively.

The form of Eqs. (67) and (68) reminds us that we have to avoid making $m_2 = 0$ and $m_1 = 0$, respectively. However, if we first multiply Eq. (67) [Eq. (68)] (and its respective inverse) by m_2 [m_1] and later we make $m_2 = 0$ [$m_1 = 0$], we can still obtain from them the confining boundary conditions (43) and (44) [(46) and (47)]. In fact, or more specifically, (*i*) [to obtain the boundary condition (43)]: from Eq. (67), we have $\phi_2(a) = 0$, but then, $m_0 = 1$. From the inverse of (67), we have $\phi_2(b) = 0$, but then $m_0 = 1$. (*ii*) [To obtain the boundary condition (44)]: from Eq. (67), we have $\phi_1(a) = 0$, but then, $m_0 = -1$. From the inverse of (67), we have $\phi_1(b) = 0$, but then, $m_0 = -1$. Likewise, (*iii*) [to obtain the boundary condition (46)]: from Eq. (68), we have $\phi_1(a) = 0$, but then, $m_3 = 1$. From the inverse of (68), we have $\phi_2(b) = 0$, but then $m_3 = 1$. (*iv*) [To obtain the boundary condition (47)]: from Eq. (68), we have $\phi_2(a) = 0$, but then, $m_3 = -1$. From the inverse of (68), we have $\phi_1(b) = 0$, but then, $m_3 = -1$. Hence, from these results, it is clear that the one-parameter families of boundary conditions (67) and (68) comprise the most general set of boundary conditions for the Majorana particle in the Majorana basis and include both the non-confining boundary conditions and the (four) confining boundary conditions.

The boundary conditions (67) and (68) can be passed to the Dirac basis via the matricial relation (48). Thus, the condition in (67) takes the form

$$\begin{bmatrix} \varphi(b) + \mathrm{i}\chi(b) \\ \chi(b) + \mathrm{i}\varphi(b) \end{bmatrix} = \frac{1}{m_2} \begin{bmatrix} 0 & -m_0 - 1 \\ m_0 - 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi(a) + \mathrm{i}\chi(a) \\ \chi(a) + \mathrm{i}\varphi(a) \end{bmatrix}, \tag{69}$$

and the condition in (68) is written as follows

$$\begin{bmatrix} \varphi(b) + \mathrm{i}\chi(b) \\ \chi(b) + \mathrm{i}\varphi(b) \end{bmatrix} = \frac{1}{m_1} \begin{bmatrix} m_3 + 1 & 0 \\ 0 & -m_3 + 1 \end{bmatrix} \begin{bmatrix} \varphi(a) + \mathrm{i}\chi(a) \\ \chi(a) + \mathrm{i}\varphi(a) \end{bmatrix}. \tag{70}$$

Using the Majorana condition in the Dirac representation, $\chi = -\mathrm{i}\varphi^*(\Leftrightarrow\ \varphi = -\mathrm{i}\chi^*)$ [Eq. (24)], we can write the boundary conditions in (69) and (70); for example, in terms of only φ, in fact

$$\begin{bmatrix} \varphi(b) \\ \varphi^*(b) \end{bmatrix} = \frac{\mathrm{i}}{m_2} \begin{bmatrix} -m_0 & 1 \\ -1 & m_0 \end{bmatrix} \begin{bmatrix} \varphi(a) \\ \varphi^*(a) \end{bmatrix}, \tag{71}$$

from Eq. (69), with $(m_0)^2 + (m_2)^2 = 1$, and

$$\begin{bmatrix} \varphi(b) \\ \varphi^*(b) \end{bmatrix} = \frac{1}{m_1} \begin{bmatrix} 1 & m_3 \\ m_3 & 1 \end{bmatrix} \begin{bmatrix} \varphi(a) \\ \varphi^*(a) \end{bmatrix}, \tag{72}$$

from Eq. (70), with $(m_1)^2 + (m_3)^2 = 1$. Following a similar procedure to that applied on conditions (67) and (68) by making $m_2 = 0$ and $m_1 = 0$, but this time on the conditions (71) and (72), respectively (and on their respective inverses), we obtain the boundary condition (53) [from Eq. (71) with $m_0 = 1$] and Eq. (54) [from Eq. (71) with $m_0 = -1$]. Likewise, we obtain the boundary condition (55) [from Eq. (72) with $m_3 = 1$] and Eq. (56) [from Eq. (72) with $m_3 = -1$]. The one-parameter families of boundary conditions (71) and (72) comprise the most general set of boundary conditions for Eq. (27), which is what models the Majorana fermion. That equation arises by imposing the Majorana condition, $\psi = \psi_C$, upon the Dirac equation in the Dirac basis.

The value of the probability current density, $j \equiv c\psi^\dagger \hat{\alpha}\, \psi$, does not depend on the representation. Thus, in one representation the value is equal to the value in any other representation. In terms of the components of ψ' (ϕ_1 and ϕ_2) and ψ (φ and χ), we can write

$$\frac{j}{c} = (\phi_1)^*\phi_2 + (\phi_2)^*\phi_1 = \varphi^*\chi + \chi^*\varphi, \tag{73}$$

and now using the Majorana condition, i.e., using the relations $\phi_1 = (\phi_1)^*$, $\phi_2 = (\phi_2)^*$ (in the Majorana basis), and $\chi = -\mathrm{i}\varphi^*$ (in the Dirac basis), we write

$$\frac{j}{c} = 2\phi_1\phi_2 = -\mathrm{i}(\varphi^*)^2 + \mathrm{i}(\varphi)^2 = \xi^\dagger \hat{\sigma}_y\, \xi, \tag{74}$$

where $\xi \equiv [\,\varphi\ \varphi^*\,]^{\mathrm{T}}$. It is easy to see that for the four confining boundary conditions in (53)-(56), the following relation is verified:

$$j(a) = j(b) = 0. \tag{75}$$

Likewise, for the not confining boundary conditions given in (71) and (72), i.e., with $m_2 \neq 0$ and $m_1 \neq 0$, respectively, we only have

$$j(a) = j(b). \tag{76}$$

These are the expected results. We note in passing that the probability current density for the Dirac single-particle [Eq. (73)] satisfies the equality $j = j_C$ at all points, where $j_C \equiv c\,\psi_C^\dagger\, \hat{\alpha}\, \psi_C$. In fact, j_C is a real-valued function; therefore, $j_C = c\,(\psi_C^*)^\dagger\, \hat{\alpha}^*\, \psi_C^* = c\,\psi^\dagger\, \hat{S}_C\, \hat{\alpha}^*\, \hat{S}_C^{-1}\psi$, where the relations (10) and (14) and the unitarity of $\hat{S}_C$ have been used. Now using the relation $\hat{S}_C\, \hat{\alpha}^*\, \hat{S}_C^{-1} = +\hat{\alpha}$, which comes from Eq. (6), we obtain the desired result. Obviously, the latter is also valid for the Majorana particle. On the other hand, the Dirac wave function ψ and its charge-conjugate ψ_C should have opposite charge current densities [22] (i.e., abandoning the Dirac's theory as a single-particle theory). To obtain a physically meaningful result in this case, we could define $J \equiv e\,j$ and $J_C \equiv -e\,j_C = -e\,j$ (i.e., these two quantities should carry opposite charges, e and $-e$); therefore, $J = -J_C$. We should also have $e = 0$ for the Majorana particle; therefore, $J = J_C = 0$ at all points.

IV. THE MAJORANA PARTICLE IN A BOX WITH THE MOST GENERAL LORENTZ POTENTIAL IN (1+1) DIMENSIONS

If in the free Dirac equation given in Eq. (1) the so-called standard minimal substitution is made, $\partial_\mu \to \partial_\mu + \frac{\mathrm{i}e}{\hbar c}A_\mu$, where $A^\mu = (A^0, A^1) = (\Phi, \mathcal{A})$ ($\Phi \in \mathbb{R}$ is the external electric potential and $\mathcal{A} \in \mathbb{R}$ is the vector potential or the spatial component of A^μ), then the Dirac equation for a fermion with charge e in the presence of some external electromagnetic field A^μ is obtained. However, the potential A^μ, that is, a Lorentz two-vector, is not the most general

potential that can be added to the free Dirac equation. The most general potential is a Lorentz covariant potential, that is, a 2×2 matrix-valued function,

$$\hat{V}_{\rm cov} = \frac{1}{\hbar c}\mathcal{S}\,\hat{1}_2 - {\rm i}\hat{\gamma}^\mu \frac{{\rm i}e}{\hbar c}A_\mu + \frac{1}{\hbar c}\mathcal{W}\,\hat{\gamma}^5, \tag{77}$$

where $\hat{\gamma}^5 \equiv {\rm i}\hat{\gamma}^0\hat{\gamma}^1 = {\rm i}\hat{\alpha} = -(\hat{\gamma}^5)^\dagger$, $\mathcal{S}\in\mathbb{R}$ is a scalar potential, and $\mathcal{W}\in\mathbb{R}$ is a pseudoscalar potential. The Dirac equation in the potential $\hat{V}_{\rm cov}$ has the form

$$\left({\rm i}\hat{\gamma}^\mu\partial_\mu - \hat{V}_{\rm cov} - \frac{{\rm m}c}{\hbar}\right)\psi = 0. \tag{78}$$

The latter can also be written in the Hamilton form. In fact, by developing the sums in Eq. (78) (with the potential given by Eq. (77)), we have

$$\left[\frac{1}{c}\hat{\gamma}^0{\rm i}\frac{\partial}{\partial t} + \hat{\gamma}^1\left({\rm i}\frac{\partial}{\partial x} + \frac{e}{\hbar c}\mathcal{A}\right) - \hat{\gamma}^0\frac{e}{\hbar c}\Phi - \frac{1}{\hbar c}\mathcal{S}\,\hat{1}_2 - \frac{1}{\hbar c}\mathcal{W}\,\hat{\gamma}^5 - \frac{{\rm m}c}{\hbar}\right]\psi = 0. \tag{79}$$

And now, we obtain the desired equation by multiplying Eq. (79) by the matrix $\hbar c\hat{\gamma}^0 (\equiv \hbar c\hat{\beta})$ from the left and using the relations $(\hat{\gamma}^0)^2 = \hat{1}_2$, $\hat{\gamma}^0\hat{\gamma}^1 = \hat{\alpha}$, and $\hat{\gamma}^0\hat{\gamma}^5 = {\rm i}\hat{\beta}\hat{\alpha}$:

$${\rm i}\hbar\frac{\partial}{\partial t}\psi = \hat{\rm h}\psi, \tag{80}$$

where the Hamiltonian operator is given by

$$\hat{\rm h} = c\hat{\alpha}\left(-{\rm i}\hbar\frac{\partial}{\partial x} - \frac{e}{c}\mathcal{A}\right) + e\Phi + \mathcal{S}\,\hat{\beta} + \mathcal{W}\,{\rm i}\hat{\beta}\hat{\alpha} + {\rm m}c^2\hat{\beta}. \tag{81}$$

This is a Hermitian operator, or formally self-adjoint, because the potentials, $\mathcal{A}$, Φ, $\mathcal{S}$, and $\mathcal{W}$, are real-valued functions; and also because ${\rm i}\hat{\beta}\hat{\alpha}$ is a Hermitian matrix, as $\hat{\alpha}$ and $\hat{\beta}$ are Hermitian. Certainly, we also formally have $\hat{\rm p} \equiv -{\rm i}\hbar\partial/\partial x = \hat{\rm p}^\dagger$.

The charge-conjugate wave function of ψ, $\psi_C \equiv \hat{S}_C\,\psi^*$, satisfies the following equation:

$$\left({\rm i}\hat{\gamma}^\mu\partial_\mu - \hat{V}^C_{\rm cov} - \frac{{\rm m}c}{\hbar}\right)\psi_C = 0, \tag{82}$$

where

$$\hat{V}^C_{\rm cov} = \frac{1}{\hbar c}\mathcal{S}^C\,\hat{1}_2 - {\rm i}\hat{\gamma}^\mu\frac{{\rm i}e}{\hbar c}A^C_\mu + \frac{1}{\hbar c}\mathcal{W}^C\,\hat{\gamma}^5. \tag{83}$$

Moreover, the matrices $\hat{\gamma}^\mu$ and $\hat{S}_C$ must satisfy the relation (6),

$$\hat{S}_C\,(-\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1} = \hat{\gamma}^\mu, \tag{84}$$

i.e., $\hat{S}_C\,({\rm i}\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1} = {\rm i}\hat{\gamma}^\mu$. Taking the complex conjugate of Eq. (78), placing the identity $\hat{1}_2 = (\hat{S}_C)^{-1}\hat{S}_C$ next to ψ^*, and multiplying it by $\hat{S}_C$ from the left, we have

$$\left[\hat{S}_C({\rm i}\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1}\partial_\mu - \hat{S}_C({\rm i}\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1}\frac{{\rm i}e}{\hbar c}A_\mu - \frac{1}{\hbar c}\mathcal{S}\,\hat{S}_C\hat{1}_2(\hat{S}_C)^{-1} - \frac{1}{\hbar c}\mathcal{W}\,\hat{S}_C(\hat{\gamma}^5)^*(\hat{S}_C)^{-1} - \frac{{\rm m}c}{\hbar}\hat{S}_C(\hat{S}_C)^{-1}\right]\hat{S}_C\,\psi^* = 0. \tag{85}$$

Using Eq. (84) and also $\hat{S}_C\,({\rm i}\hat{\gamma}^5)^*(\hat{S}_C)^{-1} = {\rm i}\hat{\gamma}^5 \Rightarrow \hat{S}_C\,(\hat{\gamma}^5)^*(\hat{S}_C)^{-1} = -\hat{\gamma}^5$ (this is so because we have ${\rm i}\hat{\gamma}^5 \equiv {\rm i}\hat{\gamma}^0\,{\rm i}\hat{\gamma}^1$), we can write Eq. (85) as follows

$$\left({\rm i}\hat{\gamma}^\mu\partial_\mu - {\rm i}\hat{\gamma}^\mu\frac{{\rm i}e}{\hbar c}A_\mu - \frac{1}{\hbar c}\mathcal{S}\,\hat{1}_2 + \frac{1}{\hbar c}\mathcal{W}\,\hat{\gamma}^5 - \frac{{\rm m}c}{\hbar}\right)\psi_C = 0, \tag{86}$$

and comparing this result with Eq. (82) (with the potential given by Eq. (83)),

$$\left({\rm i}\hat{\gamma}^\mu\partial_\mu + {\rm i}\hat{\gamma}^\mu\frac{{\rm i}e}{\hbar c}A^C_\mu - \frac{1}{\hbar c}\mathcal{S}^C\,\hat{1}_2 - \frac{1}{\hbar c}\mathcal{W}^C\,\hat{\gamma}^5 - \frac{{\rm m}c}{\hbar}\right)\psi_C = 0, \tag{87}$$

we obtain the following relations:

$$A_\mu^C = -A_\mu\,, \quad \mathcal{S}^C = \mathcal{S}\,, \quad \mathcal{W}^C = -\mathcal{W}. \tag{88}$$

In other words, the equivalence of Eqs. (78) and (82) implies that if ψ describes a fermion's state with positive energy in the potentials A_μ, $\mathcal{S}$ and $\mathcal{W}$, then ψ_C describes a fermion's state (not an antifermion's state) with negative energy in the potentials $-A_\mu$, $\mathcal{S}$ and $-\mathcal{W}$. Certainly, we are considering the Dirac theory as a single-particle theory.

Then, comparing the Dirac equations for ψ and $\psi_C\,(=\psi)$ and using the relations in (88), we obtain

$$A_\mu = 0\,, \quad \mathcal{W} = 0. \tag{89}$$

Therefore, the Dirac equation describing Majorana particles can only contain a scalar potential,

$$\left(\mathrm{i}\hat{\gamma}^\mu\partial_\mu - \frac{1}{\hbar c}\mathcal{S}\,\hat{1}_2 - \frac{\mathrm{m}c}{\hbar}\right)\psi = 0, \tag{90}$$

where ψ must satisfy Eq. (16), i.e., the Majorana condition. The corresponding Hamiltonian operator is given by

$$\hat{\mathrm{h}} = -\mathrm{i}\hbar c\,\hat{\alpha}\,\frac{\partial}{\partial x} + (\mathcal{S} + \mathrm{m}c^2)\hat{\beta} \tag{91}$$

(the invariance of this Hamiltonian under the charge-conjugation operation was mentioned recently in Ref. [23]); therefore, the Dirac equation [Eq. (80)] in the Majorana representation with the Majorana condition is precisely Eq. (21) with the replacement of $\mathrm{m}c^2 \to \mathcal{S} + \mathrm{m}c^2$. The latter is a real equation because $\mathcal{S}$ is a real function. Similarly, the Dirac equation [Eq. (80)] in the Dirac representation with the Majorana condition is precisely Eq. (26) with the substitution of $\mathrm{m}c^2 \to \mathcal{S}+\mathrm{m}c^2$. The latter equation implies that the upper component of ψ in the Dirac representation, φ, satisfies the following equation that models the Majorana particle in the presence of a scalar potential:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi = -\hbar c\frac{\partial}{\partial x}\varphi^* + (\mathcal{S} + \mathrm{m}c^2)\varphi. \tag{92}$$

Clearly, all the results given above for the Dirac particle in a box do not change by the presence of a scalar potential. See the discussion following Eq. (30). Particularly, the most general set of boundary conditions for the Majorana particle in a box in the Majorana basis is given by Eqs. (39) and (40). As we have discussed before, within the latter equations, there exist two plus two confining and several not confining boundary conditions. These boundary conditions can be passed to the Dirac basis and then written only in terms of the upper component of the Dirac wave function φ. In fact, all these boundary conditions compose the most general set of boundary conditions for Eq. (92), namely,

$$\begin{bmatrix} \varphi(b) \\ \varphi^*(b) \end{bmatrix} = \frac{\mathrm{i}}{m_2}\begin{bmatrix} -m_0 & 1 \\ -1 & m_0 \end{bmatrix}\begin{bmatrix} \varphi(a) \\ \varphi^*(a) \end{bmatrix}, \quad \begin{bmatrix} \varphi(b) \\ \varphi^*(b) \end{bmatrix} = \frac{1}{m_1}\begin{bmatrix} 1 & m_3 \\ m_3 & 1 \end{bmatrix}\begin{bmatrix} \varphi(a) \\ \varphi^*(a) \end{bmatrix}, \tag{93}$$

where $(m_0)^2 + (m_2)^2 = 1$ and $(m_1)^2 + (m_3)^2 = 1$ [Eqs. (71) and (72), respectively].

V. CONCLUSIONS

In summary, the most general family of boundary conditions for a Dirac single-particle in a one-dimensional box has four (real) parameters. However, the most general subfamily of confining boundary conditions has only two (real) parameters: a two-parameter family at $x = a$ and another family with the same two parameters at $x = b$ (This result was obtained making $\theta = 0 \Rightarrow u = s = 0$ in Eq. (4) of Ref. [5]). On the other hand, the most general set of boundary conditions for a Majorana single-particle in a one-dimensional box is composed of two families of boundary conditions, each one with a real parameter. This general set holds only four confining boundary conditions and many not confining boundary conditions. Specifically, because the equation that models the Majorana particle in the Dirac representation in the presence of the most general Lorentz potential (that, in this case, can only be a scalar potential) [Eq. (92)] is an equation for a single component wave function, i.e., for the upper component of the Dirac wave function φ (for example), we were able to write the most general set of boundary conditions to this equation, of course only in terms of φ [Eq. (93)]. In particular, the confining boundary conditions have the form $f(a) = g(b) = 0$, where f and g are precisely the functions $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$. The latter is a nice result because the entire Dirac wave function, $\psi = [\,\varphi\;\chi\,]^{\mathrm{T}}$, does not support the Dirichlet boundary condition at the ends of a one-dimensional box [4].

It is very interesting to note that the four confining boundary conditions for the Majorana particle are precisely the four boundary conditions that mathematically can arise from the general linear boundary condition used in the MIT bag model. Certainly, in the modeling of the Majorana particle with the latter boundary conditions, the Majorana condition must also be obeyed. The existence of this relation between the confining boundary conditions for the Majorana particle and those of the MIT bag model for the Dirac particle was, in some way, unexpected.

Although nobody knows if there are elementary particles that are Majorana fermions, the confinement of a different class of Majorana fermions in a finite region can be found in solid-state physics, for example, in certain quantum wires and topological superconductors [9, 24, 25]. In these systems, described in the formalism of second quantization, Majorana fermions have generally emerged as (non-fundamental) quasiparticles, which are their own antiparticles (holes), but the statistics of these objects is not fermionic but non-abelian [26]. We hope that our results, even when they were obtained within the formalism of first quantization, will find some concrete application in these attractive and also accessible systems. For example, it would be interesting to know if only the confining boundary conditions that we have found can actually be realized.

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