# SUSY QM with a complex partner potential in a one-dimensional box 

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#### Abstract

An alternative approach to the general problem of factorization of the Schrödinger Hamiltonian operator, in the framework of supersymmetric quantum mechanics (SUSY QM), is presented. We express the factorization of the partner Hamiltonians $H_{1} \leftrightarrow H_{2}$ in terms of the probability density and current for the ground state eigenfunction of $H_{1}$. This directly implies that the involved operators in the factorization be complex. However, being $H_{1}$ a real operator (self-adjoint) its partner $H_{2}$ is in general a complex operator. For a vanishing probability current, we recover the results of the standard SUSY QM. We consider the model problem of a free particle in a one-dimensional box with a non-standard PT-symmetric boundary condition from which a complex PT-symmetric partner Hamiltonian with real spectrum is obtained. © 2002 Published by Elsevier Science B.V.


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## 1. Introduction

One of the methods used for finding new exactly solvable potentials is the so-called factorization method, introduced by Schrödinger [1] and later extended by Infeld and Hull [2]. This method is particularly useful to discover different potentials with equivalent energy spectra in one-dimensional quantum mechanics, being this method a special case of an old one developed by Darboux [3,4]. The factoriza-

[^0]tion method provided a motivation for studying supersymmetric quantum mechanics (SUSY QM) [5]. Indeed, the techniques of supersymmetry are essentially equivalent to the factorization method of the Hamiltonian [6,7]. Several aspects have been studied within SUSY QM [8-19] and general articles and reviews on SUSY QM have been written [20-22] (see also the references therein). Likewise, various authors have studied several extensions of simple and familiar potentials [23-25], although the supersymmetric version of the standard model problem of a non-relativistic particle in an infinite square well (one-dimensional box), has been marginally studied [ $6,9,13,17,22$ ]. In fact, only the standard Dirichlet boundary condition has been considered in the literature.

In this Letter we present an alternative general approach to the problem of factorization of a real (selfadjoint) partner Hamiltonian operator $H_{1}$. The eigenfunctions of this operator are supposed to be complex. The differential operator $H_{1}$ is written in terms of the probability density and current for its groundstate eigenfunction. This implies that the involved operators in the factorization be also complex. Finally, in spite of being $H_{1}$ a real operator, its partner $H_{2}$ might be complex because its associated potential $V_{2}(x)$ is in general complex. Certainly, the standard approach of SUSY QM does not consider complex potentials, although various attempts in this direction have been made already [26,27], some of them in the context of the so-called PT-symmetric quantum mechanics [28].

Our results provide a new physical approach, with local observable quantities, to the general SUSY complexification procedure (recently studied by Andrianov et al. [26] and Bagchi, Mallik et al. [28]) with $\operatorname{Im} V_{1}(x)=0$ and $V_{2}(x)$ complex. For a free particle in a one-dimensional box where $V_{1}(x)=$ const, depending on the particular boundary condition, there exist real as well as complex eigenfunctions, being our approach especially useful in this last case. We examine a non-standard example of boundary condition for this system. However, it is worth noting that our principal results can also be applied to any one-dimensional self-adjoint Schrödinger Hamiltonian $H_{1}$ with complex eigenfunctions.

## 2. Factorization method

In this section we present our approach to the problem of factorization of a real (self-adjoint) partner Hamiltonian operator, say $H_{1}$. Let us define $H_{1}$ on the Hilbert space $H$ of square integrable functions on a configuration space $\Omega$
$\left(H_{1} \psi^{(1)}\right)(x)=\left(-\frac{\hbar^{2}}{2 m} \frac{d^{2}}{d x^{2}}+V_{1}(x)\right) \psi^{(1)}(x)$.
This operator is unbounded and its domain $D\left(H_{1}\right)$ is the set of functions $\psi^{(1)}(x)$ for which $\psi^{(1)}(x) \in H$ and $\left(H_{1} \psi^{(1)}\right)(x) \in H$, i.e., $\left\|\psi^{(1)}\right\|<\infty$ and $\left\|H_{1} \psi^{(1)}\right\|<\infty$, with the usual definition of the norm (also $\psi^{(1)}(x)$ and its derivative are absolutely continuous functions).

For a particle in a one-dimensional box $\Omega=$ [ $0, L]$ and additionally $\psi^{(1)}(x)$ must satisfy one of the following boundary conditions [29-31]

$$
\begin{align*}
& \binom{\psi^{(1)}(L)-i \lambda\left(\psi^{(1)}\right)^{\prime}(L)}{\psi^{(1)}(0)+i \lambda\left(\psi^{(1)}\right)^{\prime}(0)} \\
& \quad=U\binom{\psi^{(1)}(L)+i \lambda\left(\psi^{(1)}\right)^{\prime}(L)}{\psi^{(1)}(0)-i \lambda\left(\psi^{(1)}\right)^{\prime}(0)} \tag{2}
\end{align*}
$$

The primes mean differentiation with respect to $x$. The parameter $\lambda$ is inserted for dimensional reasons and the matrix $U$ belongs to $U(2)$. The potential inside the box $V_{1}(x)$ is real and bounded from below. The unitary matrix $U$ may be written as
$U=\left(\begin{array}{cc}e^{i \mu} e^{i \tau} \cos \theta & e^{i \mu} e^{i \gamma} \sin \theta \\ e^{i \mu} e^{-i \gamma} \sin \theta & -e^{i \mu} e^{-i \tau} \cos \theta\end{array}\right)$,
with $0 \leqslant \theta<\pi, 0 \leqslant \mu, \tau, \gamma<2 \pi$. It can be shown that for every wavefunction $\psi^{(1)}(x) \in D\left(H_{1}\right)$, the current density
$j^{(1)}(x)=\frac{\hbar}{m} \operatorname{Im}\left(\psi^{(1) *} \frac{d\left(\psi^{(1)}\right)}{d x}\right)$,
satisfies $j^{(1)}(0)=j^{(1)}(L)$. When $j^{(1)}(0)=j^{(1)}(L) \neq$ 0 , we have a "free" particle in a box, i.e., in the box, but not confined at all to the box. If $j^{(1)}(0)=j^{(1)}(L)=0$, we have a confined particle in a box $[32,33]$.

We assume $H_{1}$ has eigenvalues $E_{n}^{(1)}$ and eigenfunctions $\psi_{n}^{(1)}(x)$ with $n=0,1,2, \ldots$, which are explicitly known. The ground-state eigenfunction is $\psi_{0}^{(1)}(x)$ and its corresponding energy is $E_{0}^{(1)} \equiv 0$, consequently from Eq. (1) we write

$$
\begin{equation*}
\left(H_{1} \psi_{0}^{(1)}\right)(x)=\left(-\frac{\hbar^{2}}{2 m} \frac{d^{2}}{d x^{2}}+V_{1}(x)\right) \psi_{0}^{(1)}(x)=0 \tag{4}
\end{equation*}
$$

then
$V_{1}(x)=\frac{\hbar^{2}}{2 m} \frac{\left(\psi_{0}^{(1)}(x)\right)^{\prime \prime}}{\psi_{0}^{(1)}(x)}$.
This potential must be real if one wants to maintain the self-adjointness of $H_{1}$, however, the eigenfunction $\psi_{0}^{(1)}(x)$ is not always real, for example, the ground-state eigenfunctions associated to periodic and antiperiodic boundary conditions are complex [32,34]. On the other hand, for a free particle in a box, confined or not, the potential $V_{1}(x)$ is a constant.

Let us consider the polar form of the complex eigenfunction $\psi_{0}^{(1)}(x)$
$\psi_{0}^{(1)}(x)=R_{0} \exp \left(i \frac{S_{0}}{\hbar}\right)$,
where $R_{0}=R_{0}(x)$ and $S_{0}=S_{0}(x)$ are real functions. The probability density $\left|\psi_{0}^{(1)}(x)\right|^{2}$ and the probability current
$j_{0}^{(1)}(x)=\frac{\hbar}{m} \operatorname{Im}\left(\psi_{0}^{(1) *} \frac{d\left(\psi_{0}^{(1)}\right)}{d x}\right)$,
for the ground-state eigenfunction are given by
$\left|\psi_{0}^{(1)}(x)\right|^{2}=R_{0}^{2}, \quad j_{0}^{(1)}(x)=\frac{1}{m} R_{0}^{2}\left(S_{0}\right)^{\prime}$.
Let us insert into Eq. (5) the polar form of $\psi_{0}^{(1)}(x)$, then the real and imaginary parts of $V_{1}(x)$ are
$\operatorname{Re}\left[V_{1}(x)\right]=\frac{\hbar^{2}}{2 m}\left[\frac{\left(R_{0}\right)^{\prime \prime}}{R_{0}}-\frac{1}{\hbar^{2}}\left(\left(S_{0}\right)^{\prime}\right)^{2}\right]$,
and

$$
\begin{align*}
\operatorname{Im}\left[V_{1}(x)\right] & =\frac{\hbar^{2}}{2 m}\left[\frac{2}{\hbar} \frac{\left(R_{0}\right)^{\prime}}{R_{0}}\left(S_{0}\right)^{\prime}+\frac{1}{\hbar}\left(S_{0}\right)^{\prime \prime}\right] \\
& =\frac{\hbar}{2 m} \frac{1}{R_{0}^{2}}\left[R_{0}^{2}\left(S_{0}\right)^{\prime}\right]^{\prime}, \tag{9}
\end{align*}
$$

using one of the relations (7) we obtain
$\operatorname{Im}\left[V_{1}(x)\right]=\frac{\hbar}{2} \frac{\left(j_{0}^{(1)}(x)\right)^{\prime}}{R_{0}^{2}(x)}$.
As is well-known [35], if $\psi_{0}^{(1)}(x)$ satisfies the Schrödinger eigenvalue equation with a real potential $V_{1}(x)$, then the probability current $j_{0}(x)$ is constant
$j_{0}^{(1)}(x) \equiv j_{0}=\mathrm{const}$,
therefore
$\operatorname{Im}\left[V_{1}(x)\right]=0$.
Then, $V_{1}(x)$ is real, consistently with the selfadjointness of $H_{1}$
$V_{1}(x)=\frac{\hbar^{2}}{2 m} \frac{\left(R_{0}\right)^{\prime \prime}(x)}{R_{0}(x)}-\frac{m}{2} \frac{j_{0}^{2}}{R_{0}^{4}(x)}$.
More interestingly, physically the potential $V_{1}(x)$ depends on the Bohm's quantum potential $Q_{0}=$ $-\frac{\hbar^{2}}{2 m} \frac{1}{R_{0}} \frac{d^{2} R_{0}}{d x^{2}}$ and Bohmian velocity $v_{0}(x)=j_{0} / R_{0}^{2}(x)$
calculated for $\psi_{0}^{(1)}(x)[30,36]$ :
$V_{1}(x)=-Q_{0}(x)-\frac{m}{2} v_{0}^{2}(x)$.
This equation takes also the suggestive form $\frac{m}{2} v_{0}^{2}(x)+$ $V_{1}(x)+Q_{0}(x)=0$, since $E_{0}^{(1)}=0$ (indeed, this result is provided by the quantum Hamilton-Jacobi equation, e.g., see Eq. (3) in [32]). So, given the probability current and density corresponding to the ground-state eigenfunction, we may know the potential, up to a constant. Clearly, for a confined particle in a box $j_{0}=0$, so $V_{1}(x)$ is just Bohm's quantum potential
$V_{1}(x)=-Q_{0}(x)$.
The Hamiltonian $H_{1}$ defined in Eq. (1) is subsequently given by

$$
\begin{align*}
\left(H_{1} \psi^{(1)}\right)(x)=\frac{\hbar^{2}}{2 m}( & -\frac{d^{2}}{d x^{2}}+\frac{\left(R_{0}\right)^{\prime \prime}(x)}{R_{0}(x)} \\
& \left.-\frac{m^{2} j_{0}^{2}}{\hbar^{2}} \frac{1}{R_{0}^{4}(x)}\right) \psi^{(1)}(x) \tag{16}
\end{align*}
$$

which can be factorized as
$\left(H_{1} \psi^{(1)}\right)(x)=b_{( \pm)} a_{( \pm)} \psi^{(1)}(x)$,
where we have defined the following linear differential operators
$a_{( \pm)} \equiv \frac{\hbar}{\sqrt{2 m}}\left(\frac{d}{d x}-\frac{\left(R_{0}\right)^{\prime}(x)}{R_{0}(x)} \pm i \frac{m j_{0}}{\hbar} \frac{1}{R_{0}^{2}(x)}\right)$,
$b_{( \pm)} \equiv \frac{\hbar}{\sqrt{2 m}}\left(-\frac{d}{d x}-\frac{\left(R_{0}\right)^{\prime}(x)}{R_{0}(x)} \pm i \frac{m j_{0}}{\hbar} \frac{1}{R_{0}^{2}(x)}\right)$.

So, we have two possible factorizations for $H_{1}: H_{1}=$ $b_{(+)} a_{(+)}=b_{(-)} a_{(-)}$. If the typical case $j_{0}=0$ is considered, one has $a_{(+)}=a_{(-)}$and $b_{(+)}=b_{(-)}$.

A new pair of partner Hamiltonians $H_{2}$ can be constructed, that is

$$
\begin{equation*}
\left(H_{2} \psi^{(2)}\right)(x) \equiv a_{( \pm)} b_{( \pm)} \psi^{(2)}(x) \tag{19}
\end{equation*}
$$

where $\psi^{(2)}(x)$ denote the complex eigenfunctions of each $H_{2}$. We can write

$$
\begin{equation*}
\left(H_{2} \psi^{(2)}\right)(x)=\left(-\frac{\hbar^{2}}{2 m} \frac{d^{2}}{d x^{2}}+V_{2}(x)\right) \psi^{(2)}(x) \tag{20}
\end{equation*}
$$

where the pair of complex potentials $V_{2}(x)$ are

$$
\begin{align*}
\left(V_{2}\right)_{( \pm)}(x)= & -\frac{\hbar^{2}}{2 m} \frac{\left(R_{0}\right)^{\prime \prime}(x)}{R_{0}(x)}+\frac{\hbar^{2}}{m} \frac{\left(\left(R_{0}\right)^{\prime}\right)^{2}(x)}{R_{0}^{2}(x)} \\
& -\frac{m}{2} \frac{j_{0}^{2}}{R_{0}^{4}(x)} \mp i 2 \hbar j_{0} \frac{\left(R_{0}\right)^{\prime}(x)}{R_{0}^{3}(x)} \tag{21}
\end{align*}
$$

Clearly, $\left(V_{2}\right)_{( \pm)}(x)$ are complex potentials when $j_{0} \neq$ 0 and $\left(R_{0}\right)^{\prime}(x) \neq 0$, that is, for a non-confined particle in a box with a non-trivial complex groundstate eigenfunction. Consequently, being $H_{2}$ nonHermitian is obviously a non-self-adjoint operator. On the other hand, when $j_{0} \neq 0$ (and, for example, on the real line $\Omega=\mathcal{R}$ ), one does not expect that the set of states $\psi^{(2)}(x)$ always satisfy $\left\|\psi^{(2)}\right\|<\infty$ and $\left\|H_{2} \psi^{(2)}\right\|<\infty$.

The potentials $V_{1}(x)$ and $\left(V_{2}\right)_{( \pm)}(x)$ may also be written as
$V_{1}(x)=w_{( \pm)}^{2}(x)-\frac{\hbar}{\sqrt{2 m}}\left(w_{( \pm)}\right)^{\prime}(x)$,
$\left(V_{2}\right)_{( \pm)}(x)=w_{( \pm)}^{2}(x)+\frac{\hbar}{\sqrt{2 m}}\left(w_{( \pm)}\right)^{\prime}(x)$,
where
$w_{( \pm)}(x) \equiv \frac{\hbar}{\sqrt{2 m}}\left(-\frac{\left(R_{0}\right)^{\prime}(x)}{R_{0}(x)} \pm i \frac{m j_{0}}{\hbar} \frac{1}{R_{0}^{2}(x)}\right)$,
is a pair of "superpotentials". Note that $V_{1}(x)$ does not depend on the chosen superpotential $\left(w_{(+)}\right.$or $\left.w_{(-)}\right)$. Moreover, when $j_{0}=0$ one has the standard case $w_{( \pm)}(x) \equiv w(x)$. Clearly, the Eq. (22) are Ricatti type equations. It is worth noting that given the superpotentials $w_{( \pm)}(x)$, the function $R_{0}(x)$ may be indistinctly obtained from the real and imaginary parts of $w_{( \pm)}(x)$, in which case we obtain the expression

$$
\begin{align*}
C & \exp \left(\frac{\sqrt{2 m}}{\hbar} \int^{x} 2 \operatorname{Re} w_{( \pm)}(\tilde{x}) d \tilde{x}\right) \\
& = \pm \frac{\sqrt{2 m}}{m j_{0}} \operatorname{Im} w_{( \pm)}(x) \tag{24}
\end{align*}
$$

where $C$ is a constant and $j_{0} \neq 0$. An analogous relation has been recently mentioned (see Cannata et al. [28]). Within our approach to the problem, this relation naturally emerges.

In the case of standard and unbroken one-dimensional SUSY QM, the ground state of the Hamiltonian $H_{1}=b_{( \pm)} a_{( \pm)}: \psi_{0}^{(1)}(x)$, which is normalizable, is also
annihilated by the operators $a_{( \pm)}$. When $j_{0} \neq 0$, the annihilation of $\psi_{0}^{(1)}(x)$ is automatically ensured by the relation
$a_{(-)} \psi_{0}^{(1)}(x)=0$,
which follows from (6), (7) and (18). However, from these same relations we have
$a_{(+)} \psi_{0}^{(1)}(x) \neq 0$.
In this Letter we will only consider the operator $H_{1}=b_{(-)} a_{(-)}$and therefore the lower sign in the expressions (17)-(20) and (21)-(24). Although, when $j_{0}=0$ one has $a_{(+)}=a_{(-)}$and $b_{(+)}=b_{(-)}$. With the validity of relation (25), the absence of any zero energy state in the spectrum of $H_{2}=a_{(-)} b_{(-)}$is expressed, in general, by the relation $b_{(-)} \psi^{(2)}(x) \neq 0$.

Since $H_{1}=b_{(-)} a_{(-)}$and $H_{2}=a_{(-)} b_{(-)}$are intertwined by the operator $a_{(-)}: H_{2} a_{(-)}=a_{(-)} H_{1}$, the eigenfunctions of $H_{2}$, as well as their energy eigenvalues, are obtained from those of $H_{1}$ by using
$\psi_{n}^{(2)}(x)=a_{(-)} \psi_{n+1}^{(1)}(x)$,
$E_{n}^{(2)}=E_{n+1}^{(1)}, \quad n=0,1,2, \ldots$,
The solutions $a_{(-)} \psi_{n}^{(1)}(x)$ belong to the domain of the respective partner Hamiltonian operator $H_{2}$ with their respective boundary conditions. Clearly, the boundary conditions verified by each eigenfunction $\psi_{n}^{(2)}(x)$, are not necessarily satisfied by the eigenfunctions $\psi_{n}^{(1)}(x)$. Moreover, when $j_{0}=0$ and $R_{0}(x) \neq 0$ inside the box, the operator $a_{(-)}$does not generate a state of infinite norm because
$\left.\left\|a_{(-)} \psi\right\|^{2} \propto\left(\psi^{*} \frac{d \psi}{d x}-\frac{R_{0}^{\prime}}{R_{0}} \psi^{*} \psi\right)\right|_{0} ^{L}+\left\langle H_{1}\right\rangle_{\psi}$.
On the other hand, the (unnormalized) eigenfunctions of $H_{1}=b_{(-)} a_{(-)}$are obtained from those of $H_{2}=a_{(-)} b_{(-)}$, with the same eigenvalue, by using $\psi_{n+1}^{(1)}(x)=b_{(-)} \psi_{n}^{(2)}(x)$, with $n=0,1,2, \ldots$

From $H_{2} a_{(-)}=a_{(-)} H_{1}$ one infers that $b_{(-)}$intertwines in the other direction since $H_{1} b_{(-)}=b_{(-)} H_{2}$, moreover if $j_{0} \neq 0$ the formal adjoint of $b_{(-)}$[37] does not coincide with $a_{(-)}$, in fact $b_{(-)}^{+}=a_{(+)}$, nevertheless, $H_{1} b_{(-)}=b_{(-)} H_{2}$ is equal to $H_{1}\left(a_{(-)}^{*}\right)^{+}=$ $\left(a_{(-)}^{*}\right)^{+} H_{2}$, where $a_{(-)}^{*}$ represents the formal complex
conjugate of $a_{(-)}$. Note that this last relation is obtained from $H_{2} a_{(-)}=a_{(-)} H_{1}$ taking its complex conjugation and then its formal adjoint (it must be recalled that $H_{1}$ is real) but, in general, $H_{2}$ is complex.

Finally, it is worth pointing out that if $R_{0}(x)$ has a node then $w_{(-)}$has singularities (for a particle in a box this occurs, for example, when $\psi_{0}^{(1)}(x)$ satisfies the Neumann boundary condition $\left(\psi^{(1)}\right)^{\prime}(0)=$ $\left(\psi^{(1)}\right)^{\prime}(L)=0$, in fact, $\psi_{0}^{(1)}(x) \sim \cos (\pi x / L)$ has a node in $x=L / 2)$. In this case the operator $a_{(-)}$maps an eigenfunction $\psi_{n}^{(1)}(x)$ belonging to $D\left(H_{1}\right)$ out of the Hilbert space. In this case, as is well-known, the familiar "degeneracy" between the excited states of the partner Hamiltonians $H_{1}$ and $H_{2}$, with regular states, is only partially exhibited, see, for example, Refs. [17,22].

## 3. SUSY QM in a box: a non-standard exactly solvable example

In this section we consider a non-standard example which illustrates our approach to the factorization of the self-adjoint partner Hamiltonian $H_{1}$ in a box. In this example we have a complex ground state eigenfunction with $j_{0} \neq 0$ and $R_{0}(x) \neq$ const (see expression (21)), so, a complex partner potential $V_{2}(x)$ with real spectrum is obtained.

By making $\mu=\gamma=\theta=\pi / 2$ in (2), the following boundary condition is obtained
$\lambda\left(\psi^{(1)}\right)^{\prime}(0)=-i\left(\psi^{(1)}\right)(L) \neq 0$,
$\lambda\left(\psi^{(1)}\right)^{\prime}(L)=-i\left(\psi^{(1)}\right)(0) \neq 0$.
This is an example of a non-confined boundary condition (which can also be obtained in Carreau et al. [38] by making in their parameters $\beta=\gamma=1, \rho=-1$ and $\theta=\left(s+\frac{1}{2}\right) \pi, s=0,2,4, \ldots$, or in da Luz and Cheng [39] by imposing $\beta_{0}=\beta_{L}=1, \rho=-1$ and $\theta=\left(s+\frac{1}{2}\right) \pi, s=0,2,4, \ldots$, with their corresponding eigenfunctions and energy eigenvalues).

Let $V_{1}(x)$ be the constant potential inside the box

$$
\begin{equation*}
V_{1}(x)=V(x)-\frac{\hbar^{2} \pi^{2}}{2 m L^{2}}, \quad V(x)=0,0<x<L \tag{29}
\end{equation*}
$$

The Hamiltonian $H_{1}$ is invariant under the composed symmetry transformation PT. That is,
$\left(H_{1} P T \psi^{(1)}\right)(x)=\left(P T H_{1} \psi^{(1)}\right)(x)$,
but

$$
\left(P T \psi^{(1)}\right)(x)=\left(\psi^{(1)}\right)^{*}(L-x)
$$

must satisfy the boundary condition (29). Making use of this last relation and also of

$$
\begin{aligned}
\left(P T \psi^{(1)}\right)^{\prime}(x) & =\frac{d}{d x}\left(\psi^{(1)}\right)^{*}(L-x) \\
& =-\left.\frac{d}{d \tilde{x}}\left(\psi^{(1)}\right)^{*}(\tilde{x})\right|_{\tilde{x}=L-x}
\end{aligned}
$$

this invariance may be easily shown.
The complex normalized eigenfunctions of $H_{1}$ in $0 \leqslant x \leqslant L$ and the corresponding energy eigenvalues are
$\psi_{n}^{(1)}(x)=\frac{1}{\sqrt{\left(1+c_{n}^{2}\right) L}}\left(e^{i k_{n} x}+c_{n} e^{-i k_{n} x}\right)$,
$E_{n}^{(1)}=n(n+2) \frac{\hbar^{2} \pi^{2}}{2 m L^{2}}, \quad n=0,1,2, \ldots$,
where $c_{n}=\frac{\lambda k_{n}+(-1)^{n+1}}{\lambda k_{n}-(-1)^{n+1}}$ and $k_{n}=(n+1) \pi / L$.
The square root of the probability density as well as the probability current for the ground state are given by
$R_{0}(x)=\frac{1}{\sqrt{L}} \sqrt{1+\alpha \cos \left(2 k_{0} x\right)}, \quad j_{0}=\frac{\pi \hbar}{m L^{2}} \beta$,
where $k_{0}=\pi / L, \alpha \equiv \frac{\lambda^{2} k_{0}^{2}-1}{\lambda^{2} k_{0}^{2}+1}$ and $\beta \equiv \frac{2 \lambda k_{0}}{\lambda^{2} k_{0}^{2}+1}$.
From (21), the real and imaginary parts of $\left(V_{2}\right)_{(-)}(x) \equiv V_{2}(x)$ are

$$
\begin{align*}
\operatorname{Re} & {\left[V_{2}(x)\right] } \\
= & \frac{\pi^{2} \hbar^{2}}{2 m L^{2}} \\
& \quad \times \frac{\left[\alpha^{2} \sin ^{2}\left(2 k_{0} x\right)+2 \alpha^{2}-\beta^{2}+2 \alpha \cos \left(2 k_{0} x\right)\right]}{\left[1+\alpha \cos \left(2 k_{0} x\right)\right]^{2}} \tag{32}
\end{align*}
$$

$\operatorname{Im}\left[V_{2}(x)\right]=-2 \frac{\pi^{2} \hbar^{2}}{m L^{2}} \frac{\alpha \beta \sin \left(2 k_{0} x\right)}{\left[1+\alpha \cos \left(2 k_{0} x\right)\right]^{2}}$.
Here, PT-invariance of the potential $V_{2}(x)$ means $V_{2}(x)=V_{2}^{*}(L-x)$, for $x \in[0, L]$. In fact, the potential $V_{2}(x)$ is invariant under the PT-transformation
(and therefore the corresponding partner Hamiltonian $H_{2}$ in (20) has this symmetry, as well as $H_{1}$ ). Note that $\left(\operatorname{Re} V_{2}\right)(x)=\left(\operatorname{Re} V_{2}\right)^{*}(L-x)$ and $\left(\operatorname{Im} V_{2}\right)(x)=$ $-\left(\operatorname{Im} V_{2}\right)^{*}(L-x)$. Moreover, the real and imaginary parts of $V_{2}(x)$ do not have singularities.

The (unnormalized) complex eigenfunctions of $\mathrm{H}_{2}$ are obtained from (27) and (30)

$$
\begin{align*}
\psi_{n}^{(2)}(x) & =\left\{\frac{\hbar}{\sqrt{2 m}} \frac{d}{d x}+w(x)\right\} \psi_{n+1}^{(1)}(x) \\
& \propto\left\{\frac{d}{d x}+k_{0} \frac{\left[\alpha \sin \left(2 k_{0} x\right)-i \beta\right]}{\left[1+\alpha \cos \left(2 k_{0} x\right)\right]}\right\} \psi_{n+1}^{(1)}(x), \tag{34}
\end{align*}
$$

and the corresponding energy eigenvalues are obtained from (27).

It can be checked that the eigenfunctions $\psi_{n}^{(2)}(x)$ of $\mathrm{H}_{2}$ are also eigenfunctions of $P T$. For example, the first two eigenfunctions verify the relations $\left(P T \psi_{0}^{(2)}\right)(x)=-\psi_{0}^{(2)}(x)$ and $\left(P T \psi_{1}^{(2)}\right)(x)=\psi_{1}^{(2)}(x)$ (this is indeed so for all higher integer values of $n$ ). Therefore, the PT-symmetry of $H_{2}$ is unbroken assuring also the reality of the spectrum [40].

## 4. Conclusions

We have shown a different general approach to the problem of factorization of a real (self-adjoint) partner Hamiltonian operator $H_{1}$, which is factorized in terms of the probability density and current for its ground-state. In spite of being $H_{1}$ a real operator, its partner $H_{2}$ is a complex operator (but with a real spectrum) because its partner potential $V_{2}(x)$ may be complex. So, isospectrality between a real potential $\left(V_{1}(x)\right)$ and a complex one $\left(V_{2}(x)\right)$, may be realized. We have found an exactly solvable complex potential in a box, which has exactly the same energy levels than $V_{1}(x)=-\frac{\hbar^{2} \pi^{2}}{2 m L^{2}}$, except for the ground state. So, a "free particle" may be in a partnership relation with a particle in a complex non-constant potential.

We believe that our approach, which connects the factorization method with local observables, may be useful in all cases where one of the Hamiltonians is self-adjoint and the other is complex.

In a future detailed publication, new aspects about PT-symmetry with new consequences of our approach, as well as the study of SUSY QM for a one-dimensio-
nal Schrödinger particle in a box with different standard and non-standard boundary conditions, will be considered.

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