

# On the boundary conditions for the 1D Weyl-Majorana particle in a box

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**Abstract** In (1+1) space-time dimensions, we can have two particles that are Weyl and Majorana particles at the same time—1D Weyl-Majorana particles. That is, the right-chiral and left-chiral parts of the two-component Dirac wave function that satisfies the Majorana condition, in the Weyl representation, describe these particles, and each satisfies their own Majorana condition. Naturally, the nonzero component of each of these two two-component wave functions satisfies a Weyl equation. We investigate and discuss this issue and demonstrate that for a 1D Weyl-Majorana particle in a box, the nonzero components, and therefore the chiral wave functions, only admit the periodic and antiperiodic boundary conditions. From the latter two boundary conditions, we can only construct four boundary conditions for the entire Dirac wave function. Then, we demonstrate that these four boundary conditions are also included within the most general set of self-adjoint boundary conditions for a 1D Majorana particle in a box.

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<sup>†</sup>I would like to dedicate this article to the memory of Luis A. González-Díaz, friend and physicist.

## I. INTRODUCTION

The equation for a first quantized free massless Dirac single particle in (1+1) dimensions—or the one-dimensional free massless Dirac particle—has the form

$$i\hat{\gamma}^\mu \partial_\mu \Psi = 0, \quad (1)$$

where  $\Psi = \Psi(x, t)$  is a two-component wave function—a Dirac wave function,  $\partial_\mu = (c^{-1}\partial_t, \partial_x)$  (as usual), and the Dirac matrices  $\hat{\gamma}^\mu$ , with  $\mu = 0, 1$ , satisfy the relations  $\hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu = 2g^{\mu\nu} \hat{1}_2$ , where  $g^{\mu\nu} = \text{diag}(1, -1)$  ( $\hat{1}_2$  is the  $2 \times 2$  identity matrix), and  $(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^0 \hat{\gamma}^\mu \hat{\gamma}^0$  (the symbol  $^\dagger$  denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator) [1].

The so-called charge-conjugate wave function,  $\Psi_C \equiv \hat{S}_C \Psi^*$ , also satisfies Eq. (1), namely,

$$i\hat{\gamma}^\mu \partial_\mu \Psi_C = 0, \quad (2)$$

but this implies that

$$\hat{S}_C (i\hat{\gamma}^\mu)^* \hat{S}_C^{-1} = i\hat{\gamma}^\mu, \quad (3)$$

where  $\hat{S}_C$  is the charge-conjugation matrix (the superscript asterisk\* represents the complex conjugate) [2, 3]. The latter matrix can be chosen to be unitary (up to a phase factor) [1].

Let us introduce the following wave functions:

$$\Psi_\pm \equiv \frac{1}{2} (\hat{1}_2 \pm \hat{\Gamma}^5) \Psi, \quad (4)$$

where the (Hermitian) matrix  $\hat{\Gamma}^5 \equiv \hat{\gamma}^0 \hat{\gamma}^1$  is the chirality matrix, and satisfies the relations  $(\hat{\Gamma}^5)^2 = \hat{1}_2$  and  $\hat{\Gamma}^5 \hat{\gamma}^\mu + \hat{\gamma}^\mu \hat{\Gamma}^5 = \hat{0}_2$  ( $\hat{0}_2$  is the 2-dimensional zero matrix) [4, 5]. In addition,  $\hat{\Gamma}^5$  satisfies the relation  $\hat{S}_C (i\hat{\Gamma}^5)^* (\hat{S}_C)^{-1} = -i\hat{\Gamma}^5$ , and

$$\left[ \frac{1}{2} (\hat{1}_2 \pm \hat{\Gamma}^5) \right]^2 = \frac{1}{2} (\hat{1}_2 \pm \hat{\Gamma}^5), \quad \text{and} \quad \frac{1}{2} (\hat{1}_2 \pm \hat{\Gamma}^5) \frac{1}{2} (\hat{1}_2 \mp \hat{\Gamma}^5) = \hat{0}_2. \quad (5)$$

Note that, the two-component Dirac wave functions  $\Psi_+$  (which must also satisfy the relations  $\frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)\Psi_+ = \Psi_+$  and  $\frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)\Psi_+ = 0$ ) and  $\Psi_-$  (which must also satisfy the relations  $\frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)\Psi_- = \Psi_-$  and  $\frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)\Psi_- = 0$ ) are eigenstates of  $\hat{\Gamma}^5$ .  $\Psi_+$  is called the right-chiral eigenstate (eigenvalue +1) and  $\Psi_-$  the left-chiral eigenstate (eigenvalue -1). The charge conjugate of the wave functions  $\Psi_\pm$  verify that  $(\Psi_\pm)_C = (\Psi_C)_\pm$ , i.e., both  $\Psi_+$  and  $(\Psi_+)_C$

are right-chiral states, and similarly, both  $\Psi_-$  and  $(\Psi_-)_C$  are left-chiral states. This is not the case in (3+1) dimensions [5].

Now, note that by multiplying the Dirac equation in Eq. (1) by  $\frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)$  from the left, we obtain the equation

$$i\hat{\gamma}^\mu \partial_\mu \Psi_- = 0, \quad (6)$$

and similarly, multiplying Eq. (1) by  $\frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)$ , we obtain

$$i\hat{\gamma}^\mu \partial_\mu \Psi_+ = 0. \quad (7)$$

Certainly, because  $\Psi = \Psi_+ + \Psi_-$ , the latter pair of equations is equivalent to the Dirac equation. Because the charge-conjugate wave function  $\Psi_C$  also satisfies the Dirac equation, we also have two equations equivalent to the latter equation. Specifically, by multiplying the Dirac equation for  $\Psi_C$  by  $\frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)$  and  $\frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)$ , we obtain

$$i\hat{\gamma}^\mu \partial_\mu (\Psi_-)_C = 0, \quad \text{and} \quad i\hat{\gamma}^\mu \partial_\mu (\Psi_+)_C = 0, \quad (8)$$

respectively (remember that  $(\Psi_\pm)_C = \hat{S}_C \Psi_\pm^*$ ). So far,  $\Psi_-$  and  $(\Psi_-)_C$ , and  $\Psi_+$  and  $(\Psi_+)_C$ , are different wave functions, but they (all) satisfy the same two-component equation of motion.

The so-called (Lorentz-covariant) Majorana condition (see, for example, Refs. [5, 6]),

$$\Psi = \Psi_C, \quad (9)$$

imposed upon the two-component Dirac wave function  $\Psi$  gives us the following relations:

$$\Psi_+ = (\Psi_+)_C \quad \text{and} \quad \Psi_- = (\Psi_-)_C. \quad (10)$$

Thus, in (1+1) dimensions, if  $\Psi$  satisfies the Majorana condition, then both  $\Psi_+$  and  $\Psi_-$  satisfy this condition. Clearly, the pair of Eqs. (6) and (7) and the pair of restrictions in (10) describe a massless Majorana particle in (1+1) dimensions. Naturally, by imposing the Majorana condition on the equations in (8), we again obtain Eqs. (6) and (7).

In the Weyl representation, the two-component wave function and the Dirac matrices can be written as follows [5]:

$$\Psi \equiv \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \hat{\gamma}^0 = \hat{\sigma}_x, \quad \hat{\gamma}^1 = -i\hat{\sigma}_y \quad (11)$$

( $\hat{\sigma}_x$  and  $\hat{\sigma}_y$  are Pauli matrices). By substituting  $\Psi$ ,  $\hat{\gamma}^0$  and  $\hat{\gamma}^1$  from Eq. (11) into Eq. (1), we obtain two decoupled differential equations, namely,

$$\begin{aligned} i\hbar(\partial_t + c\partial_x)\varphi_1 &= 0, \\ i\hbar(\partial_t - c\partial_x)\varphi_2 &= 0. \end{aligned} \tag{12}$$

Likewise, in the Weyl representation, we have that  $\hat{\Gamma}^5 = \hat{\sigma}_z$ ,  $\Psi_+ = \frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)\Psi = [\varphi_1 \ 0]^T$ , and  $\Psi_- = \frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)\Psi = [0 \ \varphi_2]^T$ , as expected ( $T$  represents the transpose of a matrix); thus, the first of the equations in (12) can also be obtained from Eq. (7), and the second equation can be obtained from Eq. (6) (also as expected). Naturally, the wave function  $\Psi$  that describes the one-dimensional Dirac particle has two independent complex components, or two complex degrees of freedom, i.e., four real degrees of freedom. On the other hand, as said before, the Dirac equation (1) can also describe a one-dimensional massless Majorana single particle if, in addition,  $\Psi$  complies with the Majorana condition,  $\Psi = \Psi_C$ . Again, in the Weyl representation, we can write

$$\hat{S}_C = \exp(i\nu)\hat{\gamma}^0\hat{\gamma}^1 = \exp(i\nu)\hat{\sigma}_z (\propto \hat{\Gamma}^5), \tag{13}$$

where  $\nu = [0, 2\pi)$  is an arbitrary phase certainly not fixed by Eq. (3) but by us. Consequently, the Majorana condition in the form given in Eq. (10) gives us the following two (independent) relations:

$$\varphi_1 = \exp(i\nu)\varphi_1^* \quad \text{and} \quad \varphi_2 = -\exp(i\nu)\varphi_2^*. \tag{14}$$

Thus, the equation that describes the massless Majorana particle in (1+1) dimensions, in the Weyl representation, is a pair of decoupled one-component first-order equations, i.e., the pair of equations in (12), with the restrictions given in Eq. (14). In the end, the wave function  $\Psi$  that describes this kind of particle only has two real degrees of freedom (half of those of the Dirac particle, but it has the same degrees of freedom as that of the Weyl particle, as we will see below).

At this point, certain remarks are in order. The wave function  $\Psi$  for the one-dimensional massless Majorana particle must satisfy the one-dimensional Dirac equation (in the so-called chiral limit  $m = 0$ , or the massless limit) as well as the Majorana condition (thus, ultimately, we could also call it a massless (Dirac)-Majorana particle). This is true in any representation. Precisely, the most general set of boundary conditions for this particle when

it is within a box, in the Weyl representation, was presented in Ref. [5]. [In fact, the latter reference dealt with the massive Majorana particle, but the most general set of boundary conditions presented there does not depend on the value of the mass.] This set consists of two one-parametric families of (complex) boundary conditions for the Dirac wave function  $\Psi = [\varphi_1 \ \varphi_2]^T$  (we write and use them in section II). Certainly, all these boundary conditions arise when the self-adjointness condition is imposed on the Dirac Hamiltonian operator of the system, namely,  $\hat{H} = -i\hbar c \hat{\sigma}_z \partial_x = \hat{H}^\dagger$  (remember that the Dirac equation in Eq. (1) in its canonical form is  $i\hbar \partial_t \Psi = \hat{H} \Psi$ ), and inside its domain  $\mathcal{D}(\hat{H}) = \mathcal{D}(\hat{H}^\dagger)$ , we have precisely only these boundary conditions. Incidentally, in Ref. [5],  $\nu = 3\pi/2$  was specifically selected when choosing the charge-conjugation matrix  $\hat{S}_C$  (see Eq. (13)); nevertheless, this choice does not change the two families of boundary conditions.

Let us now return to the pair of equations in (12), forgetting how we obtained them. Precisely, these equations would be the (free) Weyl equations in (1+1) dimensions [7]. Each of the equations in (12) would describe a specific type of one-dimensional uncharged Weyl particle (a Weyl particle is always massless). In fact,  $\varphi_1$  and  $\varphi_2$  in Eq. (12) are transformed in two different ways under the Lorentz boost, i.e., they transform according to two inequivalent representations of the Lorentz group [5, 8]. Now, note that, if, for example, the one-component wave function  $\varphi_1$  that describes this kind of particle is complex-valued, then  $\varphi_1$  has one complex degree of freedom, i.e., two real degrees of freedom. The same is valid for the one-component wave function  $\varphi_2$ . Thus, a one-dimensional Weyl particle has the same degrees of freedom as the one-dimensional massless Majorana particle.

It can be noticed, although not without some surprise, that from the results given in Eqs. (6) and (7), as well as (10), one can introduce two other types of (relativistic) one-dimensional particles. Indeed, the equation that describes the first type of one-dimensional particle is given by Eq. (6), with its respective restriction given in Eq. (10), namely,

$$i\hat{\gamma}^\mu \partial_\mu \Psi_- = 0 \text{ and } \Psi_- = (\Psi_-)_C. \quad (15)$$

Similarly, the equation that describes the second type of one-dimensional particle is given by Eq. (7) with its respective restriction given in Eq. (10), namely,

$$i\hat{\gamma}^\mu \partial_\mu \Psi_+ = 0 \text{ and } \Psi_+ = (\Psi_+)_C. \quad (16)$$

Clearly, the two-component Dirac wave functions with definite chirality,  $\Psi_+$  and  $\Psi_-$ , each satisfy the one-dimensional Dirac equation (in the massless limit) and their own Majorana

conditions (thus, in principle, we could call these particles Dirac-Majorana particles again). However, it is only in the Weyl representation that it is explicitly shown that  $\Psi_+$  and  $\Psi_-$  have each one-only one-nonzero (complex) components, which are transformed independently under the Lorentz transformation (or the Lorentz boost). Certainly, this Lorentz transformation does not change the chirality of the wave function [8]. We mention in passing that due to their characteristics, the wave functions  $\Psi_+$  and  $\Psi_-$  are sometimes also called Weyl wave functions or said to satisfy the Weyl condition (thus, certainly, we could call the particles described by these wave functions Weyl-Majorana particles) [9]. The possibility that a wave function in (1+1) dimensions (and in other distinct space-time dimensions) can simultaneously satisfy the aforementioned Weyl condition (in even dimensions) and that of Majorana has been noted in the literature. For more details on this issue, see, for example, Ref. [9], appendix B, and Ref. [10], pp. 35-45.

Thus, from the results in (15), we can say that the first type of one-dimensional particle is completely defined by

$$i\hbar(\partial_t - c\partial_x)\varphi_2 = 0, \quad \text{with} \quad \varphi_2 = -\exp(i\nu)\varphi_2^* \quad (17)$$

(if  $\varphi_2 \in \mathbb{C}$ , then we just have here one real degree of freedom). Similarly, from the results in (16), we can say that the second type of one-dimensional particle is completely defined by

$$i\hbar(\partial_t + c\partial_x)\varphi_1 = 0, \quad \text{with} \quad \varphi_1 = \exp(i\nu)\varphi_1^* \quad (18)$$

(and again, if  $\varphi_1 \in \mathbb{C}$ , then we just have one real degree of freedom). Clearly, the one-component (Weyl) wave functions  $\varphi_1$  and  $\varphi_2$  satisfy each a one-dimensional Weyl equation and a condition that comes from imposing the Majorana condition on the entire wave function  $\Psi = [\varphi_1 \ \varphi_2]^T$ , i.e., on the wave functions  $\Psi_+ = [\varphi_1 \ 0]^T$  and  $\Psi_- = [0 \ \varphi_2]^T$ . In this way, we can now decide to call these particles Weyl-Majorana particles, i.e., two particles that are each a Weyl particle and a Majorana particle at the same time. Certainly, each Weyl-Majorana particle has half the (real) degrees of freedom of the Weyl particle as well as the Majorana particle. In the next section, we find the physically acceptable boundary conditions for this type of particle when it can only be inside a box.

## II. A 1D WEYL-MAJORANA PARTICLE IN A BOX

Let us consider a one-dimensional Weyl-Majorana particle in a box of size  $L$ , with ends, for example, at  $x = 0$  and  $x = L$ . First, we write the two Weyl equations in Eqs. (17) and (18) in their canonical forms in a single equation as follows:

$$i\hbar \partial_t \varphi_a = \hat{h}_a \varphi_a, \quad a = 1, 2, \quad (19)$$

where

$$\hat{h}_a \equiv -i\hbar c(-1)^{a-1} \partial_x \quad (20)$$

is the formally self-adjoint, or Hermitian, one-dimensional Weyl Hamiltonian operator, i.e.,  $\hat{h}_a = \hat{h}_a^\dagger$  (i.e., essentially without the specification of its domain). Clearly,  $\hat{h}_a$  is very similar to the usual nonrelativistic momentum operator (see, for example, Ref. [11]). The Hamiltonian  $\hat{h}_a$  is also a self-adjoint operator; this is essentially because its domain, i.e., the set of Weyl one-component wave functions  $\varphi_a = \varphi_a(x, t)$  in the Hilbert space of the square integrable functions  $\mathcal{H} = \mathcal{L}^2[0, L]$  on which  $\hat{h}_a$  can act ( $\equiv \mathcal{D}(\hat{h}_a) \subset \mathcal{H}$ ), includes the following general boundary condition dependent on a single parameter, namely,

$$\varphi_a(L, t) = \exp(i\theta) \varphi_a(0, t), \quad (21)$$

with  $\theta \in [0, 2\pi)$ ; in addition,  $\hat{h}_a \varphi_a \in \mathcal{H}$  [11]. Moreover, the scalar product in  $\mathcal{H}$  is denoted by  $\langle \psi_a, \chi_a \rangle \equiv \int_0^L dx \psi_a^* \chi_a$ , and the norm is  $\|\psi_a\| \equiv \sqrt{\langle \psi_a, \psi_a \rangle}$ . Precisely,  $\hat{h}_a$  satisfies the hermiticity condition, or the self-adjointness condition, namely,

$$\langle \psi_a, \hat{h}_a \chi_a \rangle = \langle \hat{h}_a \psi_a, \chi_a \rangle - i\hbar c(-1)^{a-1} [\psi_a^* \chi_a]_0^L = \langle \hat{h}_a \psi_a, \chi_a \rangle, \quad (22)$$

where we introduce the notation  $[f]_0^L \equiv f(x = L, t) - f(x = 0, t)$ , and  $\psi_a$  and  $\chi_a$  are Weyl wave functions in  $\mathcal{D}(\hat{h}_a) = \mathcal{D}(\hat{h}_a^\dagger)$ . Now, note that the Majorana condition imposed on the Weyl wave function  $\varphi_a$ , namely,  $\varphi_a = (-1)^{a-1} \exp(i\nu) \varphi_a^*$  (see Eqs. (17) and (18)), implies that  $\varphi_a^*$  must also comply the general boundary condition in Eq. (21), in which case the phase  $\exp(i\theta)$  in Eq. (21) must be real. Therefore,  $\theta = 0, \pi$ ; thus, the boundary conditions for a one-dimensional free Weyl-Majorana particle in a box can only be the periodic boundary condition,  $\varphi_a(L, t) = \varphi_a(0, t)$ , and the antiperiodic boundary condition,  $\varphi_a(L, t) = -\varphi_a(0, t)$ . Incidentally, these two boundary conditions are nonconfining boundary conditions, i.e., neither of these can cancel the probability current density at the ends of

the box. In effect, in this case, the probability current density (corresponding to the wave function  $\varphi_a$ ) is given by  $j_a \equiv (-1)^{a-1} c \varphi_a^* \varphi_a$ , where  $(\varphi_a^* \varphi_a)(x = L, t) = (\varphi_a^* \varphi_a)(x = 0, t)$  (the latter relation comes out of Eq. (22) and must be satisfied by all boundary conditions in the domain of  $\hat{h}_a$ ) [8]. Thus, we now also have the relation  $j_a(x = L, t) = j_a(x = 0, t)$ , which is obviously satisfied by the periodic and antiperiodic boundary conditions and by all boundary conditions within Eq. (21). We mention in passing that, because the solutions of the Weyl equations in (19) can always be chosen to be real, these solutions could only admit boundary conditions that are within Eq. (21) with the condition  $\varphi_a = \varphi_a^*$ , which implies that only periodic and antiperiodic boundary conditions could be imposed. The latter point was recently noted in Ref. [8].

Thus, for the Weyl-Majorana particle described by the two-component wave function  $\Psi_- = [0 \ \varphi_2]^T$ , which satisfies the results in (15), the two boundary conditions given above (for  $\varphi_2$ ) must be written as follows (we omit the variable  $t$  in the boundary conditions hereinafter):

$$\Psi_-(L) = \Psi_-(0) \quad (23)$$

(the periodic boundary condition), and

$$\Psi_-(L) = -\Psi_-(0) \quad (24)$$

(the antiperiodic boundary condition). Similarly, for the Weyl-Majorana particle described by the two-component wave function  $\Psi_+ = [\varphi_1 \ 0]^T$ , which satisfies the results in (16), the boundary conditions for  $\varphi_1$  must now be written as follows:

$$\Psi_+(L) = \Psi_+(0), \quad (25)$$

and

$$\Psi_+(L) = -\Psi_+(0) \quad (26)$$

(again, the periodic and antiperiodic boundary conditions). These are the only boundary conditions that can be imposed on  $\Psi_-$  and  $\Psi_+$  when they describe a one-dimensional Weyl-Majorana particle.

From the boundary conditions in Eqs. (23)-(26), just four boundary conditions for the Dirac wave function  $\Psi$  can be constructed. In effect, because  $\Psi = \Psi_+ + \Psi_-$ , where  $\Psi_+$  and  $\Psi_-$  are given in Eq. (4), the following results are obtained:

$$\Psi(L) = -\hat{\Gamma}^5 \Psi(0), \quad (27)$$



which comes from the conditions in Eqs. (23) and (26), and

$$\Psi(L) = \hat{\Gamma}^5 \Psi(0), \quad (28)$$

which comes from the conditions in Eqs. (24) and (25). Likewise,

$$\Psi(L) = \Psi(0), \quad (29)$$

which comes from the conditions in Eqs. (23) and (25), and

$$\Psi(L) = -\Psi(0), \quad (30)$$

which comes from the conditions in Eqs. (24) and (26). On the other hand, it can be shown (as we do below) that these four boundary conditions are in fact included within the most general set of self-adjoint boundary conditions for the one-dimensional (either massive or massless) Majorana particle enclosed in a box. In effect, this set is formed by the following two families of boundary conditions for the Dirac wave function  $\Psi = [\varphi_1 \varphi_2]^T$ , in the Weyl representation (see Eqs. (35) and (36) in Ref. [5]):

$$\Psi(L) = \frac{1}{m_2} \begin{bmatrix} -1 & -im_0 \\ -im_0 & +1 \end{bmatrix} \Psi(0), \quad (31)$$

where  $(m_0)^2 + (m_2)^2 = 1$ , and

$$\Psi(L) = \frac{1}{m_1} \begin{bmatrix} +1 & -im_3 \\ +im_3 & +1 \end{bmatrix} \Psi(0), \quad (32)$$

where  $(m_1)^2 + (m_3)^2 = 1$ ; in addition,  $m_0$ ,  $m_2$  and  $m_1$ ,  $m_3$  are real quantities (more details of boundary conditions for the problem of a Majorana particle in a box can also be found in Refs. [12] and [13]). Then, in the first subfamily above, one first notices that by setting  $m_0 = 0$ , one has that  $(m_2)^2 = 1$ , and therefore,  $m_2 = \pm 1$ . Thus, setting  $m_0 = 0$  and  $m_2 = +1$  in Eq. (31), one obtains the boundary condition in Eq. (27), and setting  $m_0 = 0$  and  $m_2 = -1$  in Eq. (31), one obtains the boundary condition in Eq. (28). Likewise, in the second subfamily above, one first notices that by setting  $m_3 = 0$ , we have that  $(m_1)^2 = 1$ , and therefore  $m_1 = \pm 1$ . Thus, setting  $m_3 = 0$  and  $m_1 = +1$  in Eq. (32), one obtains the boundary condition in Eq. (29), and setting  $m_3 = 0$  and  $m_1 = -1$  in Eq. (32), one obtains the boundary condition in Eq. (30). Thus, the boundary conditions in Eqs. (27)-(30) are also included within the most general set of self-adjoint boundary conditions for

the 1D Majorana particle in a box. Because in this case the boundary conditions for  $\Psi$  are obtained from the physically acceptable boundary conditions for  $\Psi_+$  and  $\Psi_-$ , we can say that  $\Psi = \Psi_+ + \Psi_-$  nevertheless continues to describe a massless one-dimensional Majorana particle in a box.

### III. CONCLUSIONS

Although a Majorana particle is generally considered a massive particle and a Weyl particle is always massless, in (1+1) dimensions, we can have particles that are Weyl and Majorana particles at the same time; i.e., they are 1D Weyl-Majorana particles. Thus, the upper and lower one-component wave functions of the two-component Dirac wave function  $\Psi$ , in the Weyl representation,  $\varphi_1$  and  $\varphi_2$ , each satisfy their own Weyl equation, and as we know, each belongs to a different representation of the corresponding Lorentz group. In addition, these wave functions are independent of each other, i.e., they are not related after imposing the Majorana condition on  $\Psi = [\varphi_1 \ \varphi_2]^T$ , or equivalently, on the two-component Dirac wave functions with definite chirality (or the chiral wave functions)  $\Psi_+ = [\varphi_1 \ 0]^T$  and  $\Psi_- = [0 \ \varphi_2]^T$ . In contrast, in (3+1) dimensions, the upper and lower two-component wave functions of the four-component Dirac wave function (or bispinor), in the Weyl representation, are linked by the Majorana condition [5]. Naturally, also in this case, these two two-component wave functions each satisfy their own Weyl equation (certainly, in the massless limit of the (free) Dirac equation), and each belongs to a different representation of the corresponding Lorentz group.

To recap, in (1+1) dimensions,  $\Psi_+$  and  $\Psi_-$  each satisfy the one-dimensional Dirac equation (in the massless limit) and their own Majorana condition, but only in the Weyl representation are the nonzero components of  $\Psi_+$  and  $\Psi_-$  transformed independently under the Lorentz transformation; i.e., just under this circumstance,  $\Psi_+$  and  $\Psi_-$  each describe a 1D Weyl-Majorana particle.

For a 1D Weyl-Majorana particle in a box, the chiral wave functions only admit the periodic and antiperiodic boundary conditions. This is because the one-component Weyl wave functions  $\varphi_1$  and  $\varphi_2$  only admit these two boundary conditions. From these two boundary conditions for  $\Psi_+$  and  $\Psi_-$ , just four boundary conditions can be constructed for the entire Dirac wave function  $\Psi = \Psi_+ + \Psi_-$ . Moreover, these four boundary conditions

are included within the most general set of self-adjoint boundary conditions for a 1D (either massive or massless) Majorana particle in a box. Thus, although these boundary conditions for  $\Psi$  are obtained from the boundary conditions for  $\Psi_+$  and  $\Psi_-$ ,  $\Psi = \Psi_+ + \Psi_-$  continues to describe a 1D massless Majorana particle in a box. We believe that our paper will be of interest to all who are interested in the relativistic quantum mechanics in (1+1) dimensions.

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