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On real solutions of the Dirac equation for a one-dimensional Majorana particle



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ABSTRACT

I would like to dedicate this paper to the memory of my beloved father Carmine De Vincenzo Di Fresca, who passed away unexpectedly on March 16, 2018. That day something inside of me also died.

Keywords: Relativistic quantum mechanics of a singleparticle in (1 + 1) dimensions The Dirac equation The Majorana particle The Majorana representation We construct general solutions of the time-dependent Dirac equation in (1+1) dimensions with a Lorentz scalar potential, subject to the so-called Majorana condition, in the Majorana representation. In this situation, these solutions are real-valued and describe a one-dimensional Majorana single particle. We specifically obtain solutions for the following cases: a Majorana particle at rest inside a box, a free (i.e., in a penetrable box with the periodic boundary condition), in an impenetrable box with no potential (here we only have four boundary conditions), and in a linear potential. All these problems are treated in a very detailed and systematic way. In addition, we obtain and discuss various results related to real wave functions. Finally, we also wish to point out that, in choosing the Majorana representation, the solutions of the Dirac equation with a Lorentz scalar potential can be chosen to be real but do not need to be real. In fact, complex solutions for this equation can also be obtained. Thus, a Majorana particle cannot be described only with the Dirac equation in the Majorana representation without explicitly imposing the Majorana condition.

Introduction

In 1937, Majorana posed the question of whether a particle with a spin of 1/2 could be its own antiparticle [1]. Essentially, Majorana noted that if the gamma matrices present in the free Dirac equation

$$\left(\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu} - \frac{\mathrm{m}c}{\hbar}\,\hat{1}_{4}\right)\Psi = 0\tag{1}$$

were forced to satisfy the condition $(i\hat{\gamma}^{\mu})^* = i\hat{\gamma}^{\mu}$ ($\hat{1}_4$ is the 4 × 4 identity matrix), then Eq. (1) could have real-valued solutions. Then, Majorana found for the first time a set of matrices that satisfy this condition (this is the so-called Majorana representation). In other words, in the Majorana representation, the free complex Dirac operator (i.e., the operator acting on Ψ in Eq. (1)) becomes a real Dirac operator; therefore, one can obtain real-valued solutions for the Dirac equation. Actually, in choosing the matrices in the Majorana representation, the solutions of the free Dirac equation could be chosen to be real but do not need to be real. It is found that these results are still valid when a (real) Lorentz scalar potential is included into the free Dirac equation. In fact, no other Lorentz potential can be added; for example, in (1+1) dimensions, neither a Lorentz two-vector nor a Lorentz pseudoscalar potential can be added [2]. On a side note, the Klein-Fock-Gordon and Maxwell equations are also real equations; therefore, they can accommodate real solutions. These equations, however, can also have complex solutions.

The particle described by the real-valued solution of the Dirac equation is the Majorana particle, which would be, in fact, a massive fermion that is its own antiparticle (at least in (3+1) dimensions); hence, the particle must be electrically neutral [1,3]. In (1+1) dimensions, it is understood that this particle is the one-dimensional Majorana particle. To date, no elementary particle has been identified as a Majorana particle. Among the known spin-1/2 particles, only neutrinos could be of a Majorana nature [4]. However, a different type of Majorana fermion have recently emerged within condensed matter systems as exotic quasiparticle excitations that are their own antiparticles; for instance, see [4–7] and the references within.

Thus, in the Majorana representation, the equation that describes a Majorana particle is the Dirac equation together with the reality condition of the wave function

$$\Psi = \Psi^*. \tag{2}$$

Precisely, in the Majorana representation, Ψ^* is the charge-conjugate wave function of Ψ , Ψ_C [2,7]; therefore, the reality condition given by Eq. (2) expresses the invariance of Ψ under the charge-conjugation operation, i.e.,

$$\Psi = \Psi_C. \tag{3}$$

The latter relation is what defines a Majorana particle in any representation [8] and is called the Majorana condition. Certainly, depending on the representation, the equation for the Majorana particle is

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a real or a complex system of coupled equations or a complex single equation for a single component of the Dirac wave function [2,7]. Moreover, it is important to remember that Ψ_C has the same transformation properties as Ψ under proper Lorentz transformations [8]; hence, the Majorana condition is Lorentz covariant.

Traditionally, the Dirac wave function Ψ has been physically associated with the Dirac particle, and its charge-conjugate wave function Ψ_C has been associated with its antiparticle. This is because charge conjugation is essentially an operation that changes a particle into its antiparticle (and vice versa). However, in a single-particle theory, if Ψ describes the particle's state with positive energy, for example, in a given Lorentz scalar potential, then Ψ_c describes the particle's state (not its antiparticle's state) with negative energy, in the same potential (and whose absence appears as the presence of a physical antiparticle with positive energy, according to the so-called Dirac hole theory). Apropos of this, in the Majorana representation, Ψ^* can describe that particle's state with negative energy, but the Majorana condition continues to give us a real solution for the time-dependent Dirac equation. In any case, one can always invoke the hole theory to obtain a physical picture of the negative energy states (see, for example, pp. 131-134 in Ref. [9]). However, as is well known, the latter takes us out of the Dirac single-particle theory itself (recall the unobservable infinite Dirac sea of negative energy particles, the so-called "vacuum" state).

To recap, a Majorana particle cannot be described only with the Dirac equation in the Majorana representation without imposing the Majorana condition. In the present paper, we want to emphasize this point and highlight the importance of the Majorana condition as a necessary condition to characterize a Majorana particle. We also consider that this task is absolutely pertinent; in fact, we have seen a recently published article that incorrectly assumes that a Majorana particle (in this case, subject to the linear Lorentz scalar potential) can be described just with the Dirac equation in the Majorana representation, without imposing the Majorana condition [10].

The paper is organized as follows. The results are given in Section "The Majorana particle". First, we present the wave equation that describes a (one-dimensional) Majorana particle, namely, the time-dependent Dirac equation (in (1+1) dimensions) with a Lorentz scalar potential with its solutions restricted by the Majorana condition. We use the Majorana representation; thus, the Majorana condition will ensure that the Dirac wave function is real. Then, we construct, step-by-step, the real general solutions of this equation in the following rather simple cases: a Majorana particle at rest inside a box (with no potential), a free (i.e., in a penetrable box with a periodic boundary condition), in an impenetrable box with no potential (here we only have four possible boundary conditions), and in a linear potential. All these problems are treated in a very detailed and systematic way. In addition, throughout this section we obtain and discuss various results, some of which are related to real wave functions. For example, we note that a typical wave equation of first order in the time derivative can have real solutions, but then the corresponding Hamiltonian must be purely imaginary. If the latter condition is fulfilled, and additionally if this operator is formally self-adjoint, then its mean value in a real-valued state always vanishes. The latter result can also be extended to the momentum operator. Finally, we wish to note that in this article we are particularly careful when dealing with operators. For example, we respect the essential difference that exists between a Hermitian operator (or a formally selfadjoint operator) and a self-adjoint operator. Because in quantum mechanics observables are always represented by self-adjoint operators, we take into account only the boundary conditions that are included in the domains of (unbounded) self-adjoint operators. Finally, the conclusions are presented in the last section of the paper.

The Majorana particle

The Dirac equation describing first quantized Dirac particles (in (1+1) dimensions) in a real-valued Lorentz scalar potential $\mathscr{S} = \mathscr{S}(x)$

has the form [2]

$$\left[i\hat{\gamma}^{\mu}\partial_{\mu} - \frac{1}{\hbar c}\left(\mathscr{S} + mc^{2}\right)\hat{\mathbf{l}}_{2}\right]\Psi = 0.$$
(4)

The Dirac single-particle two-component wave function $\Psi = [\phi_1 \phi_2]^T$ is generally complex-valued (the symbol T represents the transpose of a matrix). The Dirac gamma matrices $\hat{\gamma}^{\mu}$, with $\mu = 0, 1$, satisfy $\hat{\gamma}^{\mu} \hat{\gamma}^{\nu} + \hat{\gamma}^{\nu} \hat{\gamma}^{\mu} = 2g^{\mu\nu} \hat{1}_2$, where $g^{\mu\nu} = \text{diag}(1, -1)$ ($\hat{1}_2$ is the 2 × 2 identity matrix). In the Majorana representation, the Dirac gamma matrices satisfy $(i\hat{\gamma}^{\mu})^* = i\hat{\gamma}^{\mu}$, and the operator acting on Ψ in Eq. (4) is real (the raised asterisk * denotes a complex conjugate). In the latter case, the solutions of the Dirac Eq. (4) can be chosen to be real, but clearly they do not need to be real. For the Majorana representation, we choose the following matrices:

$$\hat{\gamma}^0 = \hat{\sigma}_y, \quad \hat{\gamma}^1 = -i\hat{\sigma}_z, \tag{5}$$

where $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli matrices. Obviously, the matrices $\hat{\gamma}^{0'} = \hat{\sigma}_y$ and $\hat{\gamma}^{1'} = +i\hat{\sigma}_z$, for example, can also be considered to be a Majorana representation (incidentally, the latter choice was used in Refs. [10,11]). These two Majorana representations are related by $\hat{\gamma}^{\mu'} = \hat{\sigma}_y \hat{\gamma}^{\mu} \hat{\sigma}_y$ and $[\phi'_1 \phi'_2]^T = \hat{\sigma}_y [\phi_1 \phi_2]^T$.

The Dirac Eq. (4) also describes Majorana particles if, in addition, Ψ complies, in any representation, with so-called Majorana condition

$$\Psi = \Psi_C, \tag{6}$$

where Ψ_C , the charge-conjugate wave function, also satisfies Eq. (4) (remember that the scalar potential would be independent of the charge of the particle [2]), but this implies that

$$\widehat{S}_{\mathcal{C}}(\mathbf{i}\widehat{\gamma}^{\mu})^*(\widehat{S}_{\mathcal{C}})^{-1} = \mathbf{i}\widehat{\gamma}^{\mu}, \quad \text{where} \quad \Psi_{\mathcal{C}} \equiv \widehat{S}_{\mathcal{C}}\Psi^*, \tag{7}$$

and the matrix \hat{S}_C can be chosen to be unitary (up to a phase factor). In the Majorana representation, we have $\hat{S}_C = \hat{1}_2$; therefore, $(i\hat{\gamma}^{\mu})^* = i\hat{\gamma}^{\mu}$ (as expected), and also $\Psi_C = \Psi^*$ (by virtue of Eq. (7)). Consequently, the Majorana condition takes the form

$$\Psi = \Psi^* (\Leftrightarrow \phi_1 = \phi_1^* \quad \text{and} \quad \phi_2 = \phi_2^*), \tag{8}$$

i.e., the entire Dirac wave function must be real, and it becomes the Majorana wave function.

It is worth mentioning that the Majorana condition can also be written as $\Psi = \exp(i\theta)\Psi_C$, where θ is an arbitrary phase. We could call this condition the generalized Majorana condition, and it remains covariant under a Lorentz transformation, as it must [8]. Certainly, if this version of the Majorana condition is explicitly used, every solution Ψ that describes a Majorana particle will depend on the phase θ . We can always choose $\theta = 0$, but the freedom to choose a different value for the angle θ could be convenient. Lastly, it is important to note that this generalized Majorana condition also leads to real solutions in the Majorana representation. Moreover, this condition leads to the Majorana condition in the form given in Eq. (6) for a wave function $\tilde{\Psi}$ that is physically indistinguishable from Ψ . In effect, if the generalized Majorana condition is verified, then the following relation is verified: $\tilde{\Psi} = \hat{S}_C \tilde{\Psi}^*(\equiv \tilde{\Psi}_C)$, where $\tilde{\Psi} \equiv \exp(-i\theta/2)\Psi$.

The Dirac Eq. (4) in its Hamiltonian form is

$$\left(i\hbar\,\hat{\mathbf{l}}_{2}\frac{\partial}{\partial t}-\hat{\mathbf{h}}\right)\Psi=0,\tag{9}$$

where

$$\hat{\mathbf{h}} = c\,\hat{\alpha}\,\hat{\mathbf{p}} + (\mathscr{S} + \mathbf{m}c^2)\hat{\beta} \tag{10}$$

is the formally self-adjoint, or Hermitian, Hamiltonian operator, i.e., $\hat{\mathbf{h}} = \hat{\mathbf{h}}^{\dagger}$ (in the present paper, the symbol \dagger denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator). The Dirac matrices are $\hat{\alpha} = \hat{\gamma}^0 \hat{\gamma}^1$ and $\hat{\beta} = \hat{\gamma}^0$, and $\hat{p} = -i\hbar \hat{\mathbf{1}}_2 \partial/\partial x = \hat{p}^{\dagger}$ is the formally self-adjoint, or Hermitian, Dirac momentum operator (which is, in the end, a 2 × 2 matrix). In any Majorana representation, we have that

$$(i\hat{\gamma}^{\mu})^* = i\hat{\gamma}^{\mu}$$
; therefore, $\hat{\alpha} = \hat{\alpha}^*$ and $\hat{\beta} = -\hat{\beta}^*$. The latter implies that

$$\mathbf{h} = -\mathbf{h}^{T},\tag{11}$$

and as expected, the operator acting on Ψ in Eq. (9) is real. Thus, Eq. (11) defines a Dirac Hamiltonian in the Majorana representation. In general, if one has a dynamic equation of the form $(i\partial_t - \hat{H})\Phi = 0$ $(\partial_t \equiv \partial/\partial t)$, the operator acting on Φ is real if $\hat{H} = -\hat{H}^*$, i.e., if \hat{H} is purely imaginary. For instance, this result tell us that the time-dependent Schrödinger equation cannot have real solutions, i.e., it cannot have solutions à *la* Majorana.

In the Majorana representation that we have chosen in the present article, we have that $\hat{\alpha} = \hat{\sigma}_x$ and $\hat{\beta} = \hat{\sigma}_y$. By substituting these matrices into Eq. (10) and substituting the result obtained into Eq. (9), we obtain the following real system of two coupled equations for the functions ϕ_1 and ϕ_2 :

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t} & i\hbar c\frac{\partial}{\partial x} + i(\mathscr{S} + mc^2) \\ i\hbar c\frac{\partial}{\partial x} - i(\mathscr{S} + mc^2) & i\hbar\frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(12)

(of course, you can cancel the imaginary number!). We can also prove that each component of Ψ satisfies an equation of the Klein-Fock-Gordon type, namely,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{(-1)^{j-1}}{\hbar c}\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}x} + \frac{(\mathscr{S} + \mathrm{m}c^2)^2}{\hbar^2 c^2}\right]\phi_j = 0, \quad j = 1, 2.$$
(13)

The latter two equations are only slightly different, but in the free case ($\mathscr{S} = \text{const}$), the equations are the same. In the latter situation, Eq. (13) is precisely the free Klein-Fock-Gordon equation with mass $\text{mc}^2 + \text{const.}$

Obviously, the system of equations (12) is real because it is precisely the Dirac equation in the Majorana representation. If we explicitly demand that the solutions ϕ_1 and ϕ_2 are real-valued, then $\Psi = [\phi_1 \phi_2]^T$ describes a Majorana particle, and the Majorana condition $\Psi = \Psi_C(=\Psi^*)$ is satisfied. Complex solutions for Eq. (12) can also be obtained [10], but these do not describe a Majorana particle.

Majorana particle at rest inside a box

This case will allow us to present the essential ideas of the subject in an immediate and simple way. To describe a (one-dimensional) Majorana particle at rest inside a box with no potential ($p = \mathscr{S} = 0$), the first thing we must do is solve the following Dirac equation:

$$i\hbar \hat{1}_2 \frac{d}{dt} \Psi = \hat{h} \Psi = mc^2 \hat{\sigma}_y \Psi \tag{14}$$

(see Eqs. (9) and (10)). In this case, \hat{h} is simply a 2 × 2 Hermitian constant matrix, i.e., $\hat{\mathbf{h}} = \hat{\mathbf{h}}^{\dagger}$; in fact, $\hat{\mathbf{h}}$ is also self-adjoint, because it is a bounded operator. It could be considered that this Dirac Hamiltonian operator acts on two-component column vectors Ψ that belong to the Hilbert space of the square integrable functions, $\mathscr{H} = \mathscr{L}^2(\Omega) \oplus \mathscr{L}^2(\Omega)$ (a direct sum), where Ω is a region in \mathbb{R} of width *L* (a box) that can be arbitrarily large. In general, Ψ is a function of *x* with values in \mathbb{C}^2 (the two-dimensional complex linear space); hence, the Hilbert space could also be written as $\mathscr{H} = \mathscr{L}^2(\Omega) \otimes \mathbb{C}^2$ (a tensor product). The scalar product in \mathscr{H} is denoted by $\langle \Phi, \chi \rangle \equiv \int_{\Omega} dx \, \Phi^{\dagger} \chi$ (the integrand is the \mathbb{C}^2 -scalar product of the \mathbb{C}^2 -vectors Φ and χ), and the norm is $\|\Phi\| \equiv \sqrt{\langle \Phi, \Phi \rangle}$ [12]. The eigenfunctions of \hat{h} are denoted by $\psi_{\rm p=0}^{(+)} \equiv \psi_0^{(+)}$ $\psi_{\mathbf{p}=\mathbf{0}}^{(-)}\equiv\psi_{\mathbf{0}}^{(-)}$ and for the eigenvalues $\dot{E} = +E_{p=0} \equiv +E_0 = +mc^2$ (positive-energy) and $E = -E_{p=0} \equiv -E_0$ (negative-energy), respectively:

$$\hat{\mathbf{h}}\psi_0^{(+)} = +\mathbf{m}c^2\psi_0^{(+)}, \quad \hat{\mathbf{h}}\psi_0^{(-)} = -\mathbf{m}c^2\psi_0^{(-)}.$$
 (15)

Explicitly, these eigenfunctions can be written as follows

$$\psi_0^{(+)} \equiv \sqrt{\frac{1}{L}} \, u(0) = \sqrt{\frac{1}{2L}} \begin{bmatrix} 1\\ i \end{bmatrix} \tag{16}$$

and

$$\psi_0^{(-)} = (\psi_0^{(+)})_C = (\psi_0^{(+)})^* = \sqrt{\frac{1}{L}} (u(0))^* \equiv \sqrt{\frac{1}{L}} v(0) = \sqrt{\frac{1}{2L}} \begin{bmatrix} 1\\ -i \end{bmatrix}.$$
(17)

Moreover, they satisfy the following relations of normalization and orthogonality, as expected:

$$\langle \psi_0^{(+)}, \psi_0^{(+)} \rangle = \langle \psi_0^{(-)}, \psi_0^{(-)} \rangle = 1, \quad \langle \psi_0^{(+)}, \psi_0^{(-)} \rangle = 0, \tag{18}$$

but the following \mathbb{C}^2 -orthogonality and normalization conditions for the two-component column vectors u(0) and v(0) (that are charge-conjugates of each other) must also be satisfied:

$$(u(0))^{\dagger}u(0) = (v(0))^{\dagger}v(0) \equiv \frac{E_0}{\mathrm{m}c^2} = 1, \quad (u(0))^{\dagger}v(0) = 0.$$
(19)

The most general solution of Eq. (14) at t = 0, i.e., the initial state, can be written as

$$\Psi(t=0) \equiv \Psi(0) = c_0^{(+)}(0)\psi_0^{(+)} + c_0^{(-)}(0)\psi_0^{(-)},$$
(20)

where $c_0^{(\pm)}(0) \equiv c_{p=0}^{(\pm)}(t=0)$ are two arbitrary complex constants. But if $\Psi(0)$ is required to be normalized, then the following constraint is verified:

$$|c_0^{(+)}(0)|^2 + |c_0^{(-)}(0)|^2 = 1.$$
(21)

Because, in the end, we want to describe a Majorana particle at rest, we impose upon the wave function $\Psi(0)$ the Majorana condition, i.e., $\Psi(0) = \Psi_C(0) = (\Psi(0))^*$. The latter implies that $c_0^{(-)}(0) = (c_0^{(+)}(0))^*$; therefore, the most general initial state for the Majorana particle at rest can be written as follows

$$\Psi(0) = c_0^{(+)}(0)\psi_0^{(+)} + \text{c. c.}, \quad |c_0^{(+)}(0)| = \frac{1}{\sqrt{2}},$$
(22)

where, henceforth, c. c. means a complex conjugate, i.e., the complex conjugate of the expression to the left of the symbol c. c..

The state of the system at any subsequent time t is simply

$$\Psi(t) = \hat{U}(t)\Psi(0), \tag{23}$$

where

$$\widehat{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}t\,\widehat{\mathrm{h}}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\mathrm{i}}{\hbar}t\,\widehat{\mathrm{h}}\right)^k = \exp\left(-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t\,\widehat{\sigma}_y\right)$$
(24)

is the evolution operator. Note that, in addition to being unitary, this operator is real in this case. Thus, because $\Psi(0)$ is real, then $\Psi(t)$ will also be real (in the same way, if $\Psi(0)$ would have been complex, then $\Psi(t)$ would also be complex). It is worthwhile to mention that, because \hat{h} is a matrix, $\hat{U}(t)$ admits correctly the power series expansion in Eq. (24). Precisely, by using the latter expansion when applying $\hat{U}(t)$ to $\Psi(0)$, we finally obtain from Eq. (23) the Majorana wave function at the instant *t*

$$\Psi(t) = c_0^{(+)}(0)\psi_0^{(+)}\exp\left(-i\frac{mc^2}{\hbar}t\right) + c. c. , \qquad (25)$$

with $|c_0^{(+)}(0)| = \frac{1}{\sqrt{2}}$. Clearly, the real general state in Eq. (25) is an equal-weight superposition of the positive and negative energy complex eigenstates, i.e., of two complex stationary states, for all *t*. Incidentally, if we had used the Majorana condition in its generalized form, namely, $\Psi = \exp(i\theta)\Psi_c$, then $\Psi(0)$ in Eq. (22) and $\Psi(t)$ in Eq. (25) would have depended of θ . Specifically, we would have to multiply the c.c. term in those expressions by the phase factor $\exp(i\theta)$.

It can be demonstrated that the mean value of \hat{h} in the real normalized state $\Psi(t)$ vanishes. In effect, first of all, we have that

$$\langle \hat{\mathbf{h}} \rangle_{\Psi} \equiv \langle \Psi, \, \hat{\mathbf{h}} \Psi \rangle = -\langle \Psi, \, \hat{\mathbf{h}}^* \Psi \rangle = -\langle \Psi^*, \, \hat{\mathbf{h}}^* \Psi^* \rangle = -\langle \Psi, \, \hat{\mathbf{h}} \Psi \rangle^* = -\langle \hat{\mathbf{h}} \rangle_{\Psi}^*, \tag{26}$$

where we have made use of Eq. (11). Second, just because \hat{h} is a Hermitian matrix, the following expected result is verified:

$$\begin{split} \left\langle \hat{\mathbf{h}} \right\rangle_{\Psi} &\equiv \left\langle \Psi, \, \hat{\mathbf{h}}\Psi \right\rangle = \int_{\Omega} \, d\mathbf{x} \, \Psi^{\dagger} \hat{\mathbf{h}} \Psi = \int_{\Omega} \, d\mathbf{x} \, (\hat{\mathbf{h}}\Psi)^{\dagger} \Psi = \left\langle \hat{\mathbf{h}}\Psi, \, \Psi \right\rangle \\ &= \left\langle \Psi, \, \hat{\mathbf{h}}\Psi \right\rangle^{*} = \left\langle \hat{\mathbf{h}} \right\rangle_{\Psi}^{*}, \end{split}$$

$$(27)$$

where in the penultimate step, we made use of the scalar product property $\langle \chi, \Phi \rangle = \langle \Phi, \chi \rangle^*$. Finally, from the results given in Eqs. (26) and (27), we obtain

 $\langle \hat{\mathbf{h}} \rangle_{\Psi} = 0.$ (28)

Clearly, the latter is a general result; in fact, the mean value of a formally self-adjoint operator $\hat{H} = \hat{H}^{\dagger}$ (not just a Hermitian matrix), which also satisfies the equality $\hat{H} = -\hat{H}^*$, in a real-valued state, always vanishes. This is simply because the former condition implies that $\langle \hat{H} \rangle_{\Psi} = \langle \hat{H} \rangle_{\Psi}^*, \text{ and the latter condition implies } \langle \hat{H} \rangle_{\Psi \in \mathbb{R}} = - \langle \hat{H} \rangle_{\Psi \in \mathbb{R}}^*; \text{ thus,}$ $\langle \hat{H} \rangle_{\Psi \in \mathbb{R}} = 0.$ For instance, the Dirac momentum operator $\hat{p} = -i\hbar \hat{1}_2 \partial/\partial x$ satisfies the relation $\hat{p} = -\hat{p}^*$, and because it is also a formally self-adjoint operator, we can obtain the result $\langle \widehat{p} \rangle_{\Psi \in \mathbb{R}} = 0.$ On the other hand, the Dirac or standard velocity operator, $\hat{v} \equiv c\hat{\alpha}$, is a Hermitian matrix but satisfies the relation $\hat{v}=+\hat{v}^*$ (this is always so in the Majorana representation); therefore, the mean value $\langle \widehat{v} \rangle_{\Psi \in {\rm I\!R}}$ does not have to be zero, although it is real-valued [13]. Likewise, the socalled classical velocity operator, $\hat{v}_{cl} \equiv c^2 \hat{p} \hat{h}^{-1}$ (that corresponds to the formula of classical relativistic mechanics that gives the velocity as a function of momentum and energy) [12] is a real and formally selfadjoint operator, i.e., $\hat{v}_{cl} = \hat{v}_{cl}^*$ and $\hat{v}_{cl} = \hat{v}_{cl}^{\dagger}$; therefore, its (real) expectation value in a real state, $\langle \widehat{v}_{cl} \rangle_{\Psi \in I\!\!R},$ does not have to be zero.

It is worth mentioning that in the present case (and approximately even when $p \approx 0$, i.e., $p \ll mc$), each component of $\Psi(t)$ satisfies the equation of a harmonic oscillator with frequency $\omega \equiv mc^2/\hbar$, namely (see Eq. (13)),

$$\left(\frac{d^2}{dt^2} + \omega^2\right)\phi_j = 0, \quad j = 1, 2.$$
 (29)

Solving these equations and obviously respecting the connections between $\phi_1(t)$ and $\phi_2(t)$ from the Dirac Eq. (14), we obtain

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix},$$
(30)

which is simply Eq. (23); the latter is precisely the equation that gives us the wave function $\Psi(t)$ from $\Psi(0)$ via the evolution operator. Perhaps, Majorana knew of this simple result, which could have perfectly well been called a "Majorana oscillator" [14].

Free Majorana particle in a penetrable box

To describe a free (one-dimensional) Majorana particle ($\mathscr{S} = 0$), we must first solve the following Dirac equation:

$$i\hbar \hat{1}_2 \frac{\partial}{\partial t} \Psi = \hat{h} \Psi = \left(-i\hbar c \,\hat{\sigma}_x \frac{\partial}{\partial x} + mc^2 \hat{\sigma}_y \right) \Psi \tag{31}$$

(see Eqs. (9) and (10)). We place the particle inside a wide interval $\Omega \subset \mathbb{R}$ (e.g., in a wide penetrable box of size *L*, with ends, for example, at x = 0 and x = L). The Hamiltonian is an unbounded Hermitian operator, i.e., $\hat{h} = \hat{h}^{\dagger}$; in fact, \hat{h} is also a self-adjoint operator because its domain, i.e., the set of Dirac wave functions in $\mathscr{H} = \mathscr{L}^2(\Omega) \otimes \mathbb{C}^2$ on which \hat{h} can act ($\equiv \mathscr{D}(\hat{h}) \subset \mathscr{H}$), includes the periodic boundary condition, $\Psi(L, t) = \Psi(0, t)$ (the condition that all these functions must

satisfy and that we specifically choose to use in this section), and also $\hat{h}\Psi \in \mathscr{H}$ (for example, see Ref. [2]). Precisely, \hat{h} satisfies the hermiticity condition (or, in this case, the self-adjointness condition).

$$\langle \Phi, \, \hat{h}\chi \rangle = \langle \hat{h}\Phi, \chi \rangle - i\hbar c \left[\Phi^{\dagger} \hat{\sigma}_{x}\chi \right] |_{0}^{L} = \langle \hat{h}\Phi, \chi \rangle, \tag{32}$$

where $[f]|_0^L \equiv f(L, t) - f(0, t)$, and Φ and χ are Dirac wave functions in $\mathscr{D}(\hat{\mathbf{h}}) = \mathscr{D}(\hat{\mathbf{h}}^{\dagger})$. Note that the term evaluated at the endpoints of the interval Ω vanishes because we impose the periodic boundary condition on Φ and χ . The eigenfunctions of the Hamiltonian are denoted by $\psi_p^{(+)}$ and $\psi_p^{(-)}$ for the eigenvalues $E = +E_p = +E_{-p} = +\sqrt{(\mathbf{cp})^2 + (\mathbf{mc}^2)^2} \ge \mathbf{m}$ c^2 (positive-energies) and $E = -E_p \le -\mathbf{mc}^2$ (negative-energies), respectively:

$$\hat{\mathbf{h}}\psi_{\mathbf{p}}^{(+)} = +E_{\mathbf{p}}\psi_{\mathbf{p}}^{(+)}, \quad \hat{\mathbf{h}}\psi_{\mathbf{p}}^{(-)} = -E_{\mathbf{p}}\psi_{\mathbf{p}}^{(-)}.$$
 (33)

Explicitly, these eigenfunctions can be written as follows

$$\psi_{p}^{(+)}(x) = \psi_{p}^{(+)} \equiv \sqrt{\frac{\mathrm{m}c^{2}}{E_{p}L}} u(p) \exp\left(\mathrm{i}\frac{\mathrm{p}}{\hbar}x\right) = \sqrt{\frac{1}{2L}} \left[\frac{\mathrm{i}}{\frac{\mathrm{i}E_{p}}{\mathrm{i}cp + \mathrm{m}c^{2}}}\right] \exp\left(\mathrm{i}\frac{\mathrm{p}}{\hbar}x\right)$$
(34)

and

$$\begin{split} \psi_{p}^{(-)}(x) = \psi_{p}^{(-)} &= (\psi_{-p}^{(+)})_{C} = (\psi_{-p}^{(+)})^{*} = \sqrt{\frac{\mathrm{mc}^{2}}{E_{pL}}} (u(-p))^{*} \mathrm{exp}\left(\mathrm{i}\,\frac{\mathrm{p}}{\hbar}x\right) \\ &\equiv \sqrt{\frac{\mathrm{mc}^{2}}{E_{pL}}} \, \nu\left(-\mathrm{p}\right) \mathrm{exp}\left(\mathrm{i}\,\frac{\mathrm{p}}{\hbar}x\right) = \sqrt{\frac{1}{2L}} \left[\frac{1}{\frac{-\mathrm{i}E_{p}}{\mathrm{i}cp+\mathrm{mc}^{2}}}\right] \mathrm{exp}\left(\mathrm{i}\,\frac{\mathrm{p}}{\hbar}x\right), \end{split}$$
(35)

where

$$p = \hbar \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, ...,$$
 (36)

because we use the periodic boundary condition (it is understood that $p = p_n$, but for simplicity, we do not place the subscript *n* on the letter p). Moreover, $\psi_p^{(+)}$ and $\psi_p^{(-)}$ satisfy the following expected relations of orthonormality and orthogonality:

$$\left\langle \psi_{\rm p}^{(+)}, \psi_{\rm p'}^{(+)} \right\rangle = \left\langle \psi_{\rm p}^{(-)}, \psi_{\rm p'}^{(-)} \right\rangle = \delta_{\rm pp'}, \quad \left\langle \psi_{\rm p}^{(+)}, \psi_{\rm p'}^{(-)} \right\rangle = 0,$$
(37)

where $\delta_{pp'}$ is the Kronecker delta. Again, the mutually charge-conjugate vectors, u(p) and v(p), must satisfy \mathbb{C}^2 -orthogonality and normalization relations, namely,

$$(u(\mathbf{p}))^{\dagger}u(\mathbf{p}) = (v(-\mathbf{p}))^{\dagger}v\left(-\mathbf{p}\right) \equiv \frac{E_{\mathbf{p}}}{\mathbf{m}c^{2}} = 1, \quad (u(\mathbf{p}))^{\dagger}v\left(-\mathbf{p}\right) = 0.$$
(38)

Thus, the free particle's states $\psi_{\rm p}^{(+)}(x)$ and $\psi_{\rm p}^{(-)}(x)$ are orthonormal eigenstates of the operators \hat{p} and \hat{h} , with eigenvalues p and E_{p} and p and $-E_{\rm p}$, respectively. However, according to the Dirac hole theory (i.e., abandoning for a moment Dirac's theory as a single-particle theory), the state $\psi_{\pm p}^{(-)}(x)$ can be physically associated with the corresponding antiparticle, and it would have momentum \mp p and energy + E_p . Certainly, the operator \hat{p} for a Dirac particle in a box is also self-adjoint when the boundary condition $\Psi(L, t) = \Psi(0, t)$ is included in its own domain [13,15]. The states $\psi_{\rm p}^{(+)}(x)$ and $\psi_{\rm p}^{(-)}(x)$ are also eigenstates of $\hat{v}_{cl} \equiv c^2 \, \hat{p} \, \hat{h}^{-1}$ with eigenvalues + $c^2 p/E_p$ and - $c^2 p/E_p$, respectively. Let us note in passing that, for positive energies, if $p \to \pm \infty$, then $v_{cl} \to \pm c$; and for negative energies, if $p \to \pm \infty$, then $v_{cl} \to \mp c$, i.e., the spectrum of \hat{v}_{cl} is the interval [-c, +c]. Thus, this operator is certainly bounded and self-adjoint [12]. Notice finally that the periodic boundary condition did not provide non-trivial eigensolutions for the range of energies $-mc^2 < E < +mc^2$.

The most general wave function at t = 0 can be written as

$$\Psi(x, 0) = \sum_{p=-\infty}^{+\infty} [c_p^{(+)}(0)\psi_p^{(+)}(x) + c_p^{(-)}(0)\psi_p^{(-)}(x)]$$

= $\sum_{p=-\infty}^{+\infty} [c_p^{(+)}(0)\psi_p^{(+)}(x) + c_{-p}^{(-)}(0)\psi_{-p}^{(-)}(x)],$ (39)

where in the last expression we used the fact that $\sum_p f_p = \sum_p f_{-p}$. In addition, $c_p^{(\pm)}(0) \equiv c_p^{(\pm)}(t=0)$ are arbitrary complex constants. The latter quantities satisfy a constraint, namely,

$$\sum_{p=-\infty}^{+\infty} \left[|c_p^{(+)}(0)|^2 + |c_p^{(-)}(0)|^2 \right] = \sum_{p=-\infty}^{+\infty} \left[|c_p^{(+)}(0)|^2 + |c_{-p}^{(-)}(0)|^2 \right] = 1,$$
(40)

provided that $\Psi(x, 0)$ is a normalized wave function. Again, in a hole-theoretic description, the destruction (or removal from the "vacuum") of a negative-energy particle of momentum – p appears as the creation of its corresponding positive-energy antiparticle with momentum + p (a hole in the "vacuum"). We also write the sum in Eq. (39) in terms of $\psi_{-p}^{(-)}(x)$ just to highlight this point. In this way, we can interpret $c_{-p}^{(-)}(0)$ ($c_{p}^{(+)}(0)$) as the amplitude for an antiparticle (a particle) state with momentum + p and energy + $E_{\rm p}$. These complex-valued objects could be redefined as $c_{\rm p}^{(+)}(0) \equiv b_{\rm p}$ and $c_{\rm p}^{(-)}(0) \equiv d_{\rm p}^*$, and in the formalism of second quantization, they become time-independent operators, namely, $b_{\rm p} \rightarrow \hat{b}_{\rm p}$ (the annihilation or absorption operator for the particle) and $d_{\rm p}^* \rightarrow \hat{d}_{\rm p}^{\dagger}$ (the creation operator for the antiparticle). In particle physics, it is customary to use these symbols.

Once again, because we are describing a Majorana particle, the initial state must obey the Majorana condition, $\Psi(x, 0) = \Psi_C(x, 0) = (\Psi(x, 0))^*$. With the latter condition imposed upon $\Psi(x, 0)$ (Eq. (39)), we obtain the result $c_{-p}^{(-)}(0) = (c_p^{(+)}(0))^*$, i.e., $d_p^* = (b_p)^* \Rightarrow b_p = d_p$. Therefore, the most general initial wave function for the free Majorana particle in this penetrable box takes the form

$$\Psi\left(x,\,0\right) = \sum_{p=-\infty}^{+\infty} \left[c_p^{(+)}(0)\psi_p^{(+)}(x) + c.\,c.\right], \quad \sum_{p=-\infty}^{+\infty} |c_p^{(+)}(0)|^2 = \frac{1}{2}.$$
(41)

It is then clear that here we need only one type of constant, b_p or d_p . Abandoning the language of the single-particle theory itself, we could say that a Majorana particle is a superposition of a particle and its antiparticle, but this particle and antiparticle coincide, as expected.

The wave function $\Psi(x, t)$ can be obtained by applying the evolution operator $\widehat{U}(t) = \exp\left(-\frac{i}{\hbar}t\,\widehat{h}\right)$ to the initial wave function $\Psi(x, 0)$, but only formally. In fact, in this case, we cannot use the power series expansion of the exponential, because \widehat{h} is an unbounded operator, i.e., it cannot act on all the wave functions of the Hilbert space of the system. However, the correct result is the same as that obtained following the usual prescription (for example, see Ref. [16]), namely,

$$\Psi\left(x,\,t\right) = \sum_{p=-\infty}^{+\infty} \left[c_p^{(+)}(0)\psi_p^{(+)}(x)\exp\left(-i\frac{E_p}{\hbar}t\right) + c.\,c.\,\right],\tag{42}$$

with $\sum_{p=-\infty}^{+\infty} |c_p^{(+)}(0)|^2 = \frac{1}{2}$. Notice again that $\widehat{U}(t)$ is a real operator and therefore preserves the real character of the initial wave function. Additionally, notice that we would have to multiply the c.c. term in Eqs. (41) and (42) by exp(i θ) in the case of using the generalized Majorana condition. In the end, the real wave function in Eq. (42) can also be written as follows:

$$\Psi\left(x,\,t\right) = \sqrt{\frac{1}{L}} \sum_{p=-\infty}^{+\infty} \sqrt{\frac{mc^2}{E_p}} \left[b_p u(p) \exp\left(i\frac{p}{\hbar}x - i\frac{E_p}{\hbar}t\right) + c.\,c. \right],\tag{43}$$

where $\sum_{p=-\infty}^{+\infty} |b_p|^2 = \frac{1}{2}$. In the second quantized theory, the corresponding Majorana quantum field operator is Hermitian, i.e., $\Psi(x, t) \rightarrow \hat{\Psi}(x, t) = \hat{\Psi}^{\dagger}(x, t)$. This is because the wave function $\Psi(x, t)$,

i.e., the classical field, is real.

It is worth mentioning that the boundary condition we have used in this section, i.e., the periodic boundary condition, is just one of the infinite non-confining boundary conditions that a (one-dimensional) Majorana particle can support when it is inside a box, namely,

$$\Psi\left(L,\,t\right) = -\frac{(im_0\,\hat{\sigma}_y\,+\,\hat{\sigma}_x)}{m_2}\,\Psi\left(0,\,t\right),\tag{44}$$

with $m_2 \neq 0$ (and thus Eq. (44) is a non-confining family of boundary conditions) and $(m_0)^2 + (m_2)^2 = 1$. In addition,

$$\Psi\left(L, t\right) = \frac{m_3 \hat{\sigma}_z + \hat{l}_2}{m_1} \Psi\left(0, t\right),\tag{45}$$

with $m_1 \neq 0$ (and thus Eq.(45) is a non-confining family of boundary conditions) and $(m_1)^2 + (m_3)^2 = 1$ (see Eqs.(67) and (68) in Ref. [2]). Specifically, the periodic boundary condition, $\Psi(L, t) = \Psi(0, t)$, can be obtained from Eq.(45) by making $m_1 = 1$ and $m_3 = 0$.Certainly, none of the boundary conditions in Eqs.(44) and (45) cancels the probability current density at the ends of the box (which is why we call them non-confining boundary conditions) [2].Naturally, all these boundary conditions cancel the boundary term in Eq.(32) (this is because when all of them are included in $\mathcal{D}(h)$, the relation $\mathscr{D}(\hat{\mathbf{h}}) = \mathscr{D}(\hat{\mathbf{h}}^{\dagger})$ with $\hat{\mathbf{h}} = \hat{\mathbf{h}}^{\dagger}$ is satisfied) [2]. Therefore, if we impose $\Phi = \chi = \Psi$ in Eq.(32), we obtain, as before, the expression $\langle \Psi, \hat{h}\Psi \rangle \equiv \langle \hat{h} \rangle_{\Psi} = \langle \hat{h}\Psi, \Psi \rangle = \langle \Psi, \hat{h}\Psi \rangle^* = \langle \hat{h} \rangle_{\Psi}^*$. The latter result together with Eq.(26) (which is verified whenever Ψ is real-valued) leads to the result given in Eq.(28), i.e., $\langle \hat{h} \rangle_{\Psi} = 0$. Clearly, the boundary conditions in Eqs. (44) and (45) are real boundary conditions when the wave function describes a Majorana particle, i.e., when the wave function is real-valued. Lastly, the results obtained in this subsection obviously apply to those of subsection "Majorana particle at rest inside a box" if we only maintain the eigenvalue p = 0.

Free Majorana particle in an impenetrable box

To describe a free (one-dimensional) Majorana particle ($\mathcal{S} = 0$) that is inside an impenetrable box of width L, we must first solve Eq. (31). In this case, the self-adjoint (Dirac) Hamiltonian operator has a domain that includes a general two-real-parameter family of confining boundary conditions, i.e., a two-real-parameter family at one end of the box, and another family with the same two parameters at the other end [17]. These boundary conditions cancel the probability current density at the ends of the box (which is why we call them confining boundary conditions) [2,17]. Likewise, these boundary conditions imposed upon Φ and χ in Eq. (32) lead to the vanishing of the boundary term present there (this is simply because \hat{h} , with these boundary conditions within its domain, is a self-adjoint operator). As it was demonstrated in Ref. [2], from all the boundary conditions that confine a (one-dimensional) Dirac particle inside a box, only four of them can be used to confine a (one-dimensional) Majorana particle (due to the Majorana condition), namely,

$$\phi_2(L, t) = \phi_2(0, t) = 0, \tag{46}$$

and

$$\phi_1(L, t) = \phi_1(0, t) = 0. \tag{47}$$

These two boundary conditions are the Dirichlet boundary condition imposed upon the lower component (Eq. (46)) and the upper component of the wave function (Eq. (47)). Likewise,

$$\phi_2(L, t) = \phi_1(0, t) = 0, \tag{48}$$

and

$$\phi_1(L, t) = \phi_2(0, t) = 0. \tag{49}$$

These are two mixed boundary conditions. The authors of Ref. [11] also

obtained the result that there are just four confining boundary conditions for a (one-dimensional) Majorana particle in a box (see Eq. (3.8) therein), but they expressed these boundary conditions only in terms of a single component of the Dirac wave function in the Dirac representation. Note that, by taking the limit $m_2 \rightarrow 0$ in Eq. (44) and its respective inverse, we obtain the results in Eq. (46) (when $m_0 = 1$) and Eq. (47) (when $m_0 = -1$). Likewise, by taking the limit $m_1 \rightarrow 0$ in Eq. (45) and its respective inverse, we obtain the results in Eq. (48) (when $m_3 = 1$) and Eq. (49) (when $m_3 = -1$) [2]. Thus, the two one-parameter families of boundary conditions in Eqs. (44) and (45) can also generate the four confining boundary conditions. Consequently, these two oneparameter families really make up the most general set of boundary conditions for the (one-dimensional) Majorana particle in a box. Unexpectedly, the four confining boundary conditions for the (one-dimensional) Majorana particle are exactly the four boundary conditions that mathematically can arise from the general linear boundary condition used in the MIT bag model for a hadronic structure (certainly, the four boundary conditions in this model must also obey the Majorana condition) [2], and the boundary condition in Eq. (49) is precisely the boundary condition commonly used in that model, but in one dimension [2].

Once again, we use the following notation to identify the normalized eigenfunctions of the Hamiltonian in Eq. (31): $\psi_p^{(+)}$ and $\psi_p^{(-)}$, for the eigenvalues $E = +E_p = +E_{-p} = +\sqrt{(cp)^2 + (mc^2)^2} \ge mc^2$ (positive-energies) and $E = -E_p \le -mc^2$ (negative-energies), respectively (see Eq. (33)). Likewise, we write $\psi_q^{(+)}$ and $\psi_q^{(-)}$ for the eigenvalues $E = +E_q = +E_{-q} = +\sqrt{(mc^2)^2 - (cq)^2} < mc^2$ (positive-energies) and $E = -E_q > -mc^2$ (negative-energies), respectively. The latter eigenfunctions can be obtained from the former with the replacement $p \rightarrow -iq$, where q > 0; thus, $\psi_p^{(\pm)} \rightarrow \psi_q^{(\pm)}$, and $E_p \rightarrow E_q$.

Naturally, for each of the four confining boundary conditions, we will have a different orthonormal system of eigenstates of \hat{h} . However, all these eigenstates will not be eigenstates of the Dirac momentum operator $\hat{p} = -i\hbar \hat{1}_2 \partial/\partial x$; thus, the letter p in the eigenfunctions will just be a label. Moreover, the operator \hat{p} in an interval is self-adjoint just when $\Psi(L, t) = \hat{M}\Psi(0, t)$, where \hat{M} is a unitary matrix, but the confining boundary conditions in Eqs. (46)–(49) are not included in that latter general boundary condition [13,15]. Thus, in the present problem, we do not have a completely (physically) acceptable momentum operator. However, if we could instantly remove the walls of the box and immediately measure the momentum, that would put the particle in an eigenstate of the momentum operator, which would be, in this case, a self-adjoint operator with a continuous spectrum (see, for example, p. 269 in Ref. [18]). Explicitly, the eigenfunctions of the Hamiltonian can be written as follows.

For $\phi_2(L, t) = \phi_2(0, t) = 0$,

$$\psi_{p}^{(+)}(x) = \psi_{p}^{(+)} = \sqrt{\frac{1}{L}} \begin{bmatrix} \frac{cp}{E_{p}} \cos\left(\frac{px}{\hbar}\right) + \frac{mc^{2}}{E_{p}} \sin\left(\frac{px}{\hbar}\right) \\ isin\left(\frac{px}{\hbar}\right) \end{bmatrix}$$
(50)

and

$$\psi_{p}^{(-)}(x) = \psi_{p}^{(-)} = (\psi_{-p}^{(+)})_{C} = (\psi_{-p}^{(+)})^{*} = \sqrt{\frac{1}{L}} \begin{bmatrix} -\frac{cp}{E_{p}} \cos\left(\frac{px}{\hbar}\right) - \frac{mc^{2}}{E_{p}} \sin\left(\frac{px}{\hbar}\right) \\ isin\left(\frac{px}{\hbar}\right) \end{bmatrix}$$
(51)

where

$$p = \hbar \frac{\pi N}{L}, \quad N = 1, 2, 3,$$
 (52)

For $\phi_1(L, t) = \phi_1(0, t) = 0$,

$$\psi_{p}^{(+)}(x) = \psi_{p}^{(+)} = \sqrt{\frac{1}{L}} \begin{bmatrix} \operatorname{isin}\left(\frac{px}{\hbar}\right) \\ \frac{cp}{E_{p}} \cos\left(\frac{px}{\hbar}\right) - \frac{mc^{2}}{E_{p}} \sin\left(\frac{px}{\hbar}\right) \end{bmatrix},$$
(53)

and

$$\psi_{p}^{(-)}(x) = \psi_{p}^{(-)} = (\psi_{-p}^{(+)})_{\mathcal{C}} = (\psi_{-p}^{(+)})^{*} = \sqrt{\frac{1}{L}} \begin{bmatrix} \sin\left(\frac{px}{\hbar}\right) \\ -\frac{cp}{E_{p}}\cos\left(\frac{px}{\hbar}\right) + \frac{mc^{2}}{E_{p}}\sin\left(\frac{px}{\hbar}\right) \end{bmatrix},$$
(54)

where

$$p = \hbar \frac{\pi N}{L}, \quad N = 1, 2, 3,$$
 (55)

Notice that these two boundary conditions do not provide non-trivial eigensolutions in the range of energies $-mc^2 \le E \le +mc^2$.

For $\phi_2(L, t) = \phi_1(0, t) = 0$, in this case, the eigenfunctions of the Hamiltonian for the energies $E > mc^2$ and $E < -mc^2$ are precisely $\psi_p^{(+)}$ and $\psi_p^{(-)}$, respectively, given by Eqs. (53) and (54). However, the spectrum of $\hat{\mathbf{h}}$ is obtained from the following transcendental equation

$$\frac{\hbar}{\mathrm{mc}}\frac{1}{L}\frac{\mathrm{pL}}{\hbar} = \mathrm{tan}\left(\frac{\mathrm{pL}}{\hbar}\right),\tag{56}$$

whose solutions must obey the inequality p > 0. The eigensolution for p = 0 ($\Rightarrow E = \pm mc^2$) is the trivial solution whenever $L \neq \hbar/mc$. Moreover, we have one positive-energy eigenfunction,

$$\psi_{q}^{(+)}(x) = \psi_{q}^{(+)} = \sqrt{\frac{1}{L}} \begin{bmatrix} \sinh\left(\frac{qx}{\hbar}\right) \\ -\frac{icq}{E_{q}}\cosh\left(\frac{qx}{\hbar}\right) + \frac{imc^{2}}{E_{q}}\sinh\left(\frac{qx}{\hbar}\right) \end{bmatrix},$$
(57)

with energy $E = +E_q$, and the corresponding negative-energy eigensolution,

$$\begin{split} \psi_{\mathbf{q}}^{(-)}(\mathbf{x}) &= \psi_{\mathbf{q}}^{(-)} = \left(\psi_{-\mathbf{q}}^{(+)}\right)_{C} = \left(\psi_{-\mathbf{q}}^{(+)}\right)^{*} \\ &= \left(-1\right) \sqrt{\frac{1}{L}} \left[\frac{\sinh\left(\frac{q\mathbf{x}}{h}\right)}{\frac{\mathrm{i}c\mathbf{q}}{E_{\mathbf{q}}} \mathrm{cosh}\left(\frac{q\mathbf{x}}{h}\right) - \frac{\mathrm{i}mc^{2}}{E_{\mathbf{q}}} \mathrm{sinh}\left(\frac{q\mathbf{x}}{h}\right)} \right], \end{split}$$
(58)

with energy $E = -E_q$. The quantity q is the only positive solution of the equation

$$\frac{\hbar}{\mathrm{mc}}\frac{1}{L}\frac{\mathrm{q}L}{\hbar} = \tanh\left(\frac{\mathrm{q}L}{\hbar}\right),\tag{59}$$

where $L > \hbar/mc$.

For $\phi_1(L, t) = \phi_2(0, t) = 0$, in this case, the eigenfunctions of the Hamiltonian are precisely $\psi_p^{(+)}$ and $\psi_p^{(-)}$ given by Eqs. (50) and (51). Again, the spectrum of $\hat{\mathbf{h}}$ is obtained from a transcendental equation, namely,

$$-\frac{\hbar}{\mathrm{mc}}\frac{1}{L}\frac{\mathrm{p}L}{\hbar} = \mathrm{tan}\left(\frac{\mathrm{p}L}{\hbar}\right),\tag{60}$$

and its solutions must satisfy the inequality p > 0. In this case, the eigensolution for p = 0 ($\Rightarrow E = \pm mc^2$) is just the trivial solution. Likewise, we do not have non-trivial eigensolutions in the range of energies $-mc^2 < E < +mc^2$.

As expected, the eigenvectors corresponding to each boundary condition satisfy the relations of orthonormality and orthogonality identical to those given in Eq. (37). Additionally, the states (57) and (58) verify the following relations

$$\left\langle \psi_{\mathbf{q}}^{(+)},\psi_{\mathbf{q}}^{(+)}\right\rangle = \left\langle \psi_{\mathbf{q}}^{(-)},\psi_{\mathbf{q}}^{(-)}\right\rangle = 1, \quad \left\langle \psi_{\mathbf{q}}^{(+)},\psi_{\mathbf{q}}^{(-)}\right\rangle = 0, \tag{61}$$

and also

$$\left\langle \psi_{\mathbf{q}}^{(+)}, \psi_{\mathbf{p}}^{(+)} \right\rangle = \left\langle \psi_{\mathbf{q}}^{(+)}, \psi_{\mathbf{p}}^{(-)} \right\rangle = \left\langle \psi_{\mathbf{q}}^{(-)}, \psi_{\mathbf{p}}^{(+)} \right\rangle = \left\langle \psi_{\mathbf{q}}^{(-)}, \psi_{\mathbf{p}}^{(-)} \right\rangle = 0,$$
(62)

for all p that is solution of Eq. (56).

Again, the most general normalized Dirac wave function at t = 0 can be written similarly to that in Eq. (39) and is subject to a restriction similar to that in Eq. (40), namely,

$$\Psi\left(x,0\right) = c_{q}^{(+)}(0)\psi_{q}^{(+)}(x) + c_{q}^{(-)}(0)\psi_{q}^{(-)}(x) + \sum_{p}^{+\infty} \left[c_{p}^{(+)}(0)\psi_{p}^{(+)}(x) + c_{p}^{(-)}(0)\psi_{p}^{(-)}(x)\right],$$
(63)

and

$$|c_{q}^{(+)}(0)|^{2} + |c_{q}^{(-)}(0)|^{2} + \sum_{p}^{+\infty} [|c_{p}^{(+)}(0)|^{2} + |c_{p}^{(-)}(0)|^{2}] = 1,$$
(64)

respectively (also, $c_q^{(\pm)}(0) \equiv c_q^{(\pm)}(t=0)$ and $c_p^{(\pm)}(0) \equiv c_p^{(\pm)}(t=0)$, as before). Certainly, in each case, the infinite sum is determined by the relations given in Eqs. (52), (55), (56) and (59). Moreover, when considering the boundary conditions (46), (47) and (49), only the terms with sums should be written in Eqs. (63) and (64), i.e., the terms depending on q are not present in those cases. The initial state for a Majorana particle must satisfy the Majorana condition, $\Psi(x, 0) = \Psi_C(x, 0) = (\Psi(x, 0))^*$. The latter condition together with the relations $(\psi_p^{(\pm)})^* = \psi_{-p}^{(\mp)} = -\psi_p^{(\mp)}$ and $(\psi_q^{(\pm)})^* = \psi_{-q}^{(\mp)} = -\psi_q^{(\mp)}$ (in the last step of each of these equalities, we use the fact that all these eigenfunctions are odd in p and q) leads to the relations $c_p^{(-)}(0) = -(c_p^{(+)}(0))^*$ and $c_q^{(-)}(0) = -(c_q^{(+)}(0))^*$. Substituting the latter relations in Eq. (63), as well as those given before for the eigenfunctions, we can write

$$\Psi\left(x,\,0\right) = \left[c_{q}^{(+)}(0)\psi_{q}^{(+)}(x) + c.\,c.\right] + \sum_{p}^{+\infty} \left[c_{p}^{(+)}(0)\psi_{p}^{(+)}(x) + c.\,c.\right],$$
(65)

with $|c_q^{(+)}(0)|^2 + \sum_p^{+\infty} |c_p^{(+)}(0)|^2 = \frac{1}{2}$.

Again, the real general solution $\Psi(x, t)$ is simply given by

$$\Psi\left(x, t\right) = \left[c_{q}^{(+)}(0)\psi_{q}^{(+)}(x)\exp\left(-i\frac{E_{q}}{\hbar}t\right) + c. c.\right] + \sum_{p}^{+\infty} \left[c_{p}^{(+)}(0)\psi_{p}^{(+)}(x)\exp\left(-i\frac{E_{p}}{\hbar}t\right) + c. c.\right],$$
(66)

with the restriction that follows Eq. (65), which comes from the normalization of the initial state and that always remains equal to one. The correct result given by Eq. (66) can be obtained by formally applying the evolution operator to the wave function $\Psi(x, 0)$. In fact, the four Hamiltonian operators considered here act in the manner presented in Eq. (31) but do not have equal domains because each of these has a different boundary condition. Certainly, they are unbounded operators, by acting only on a subset of the Hilbert space.

A Majorana particle in a linear Lorentz scalar potential

In this case, we choose $\mathcal{S} = kx$, where k > 0 is a constant. The Dirac equation we must solve is

$$i\hbar \hat{1}_2 \frac{\partial}{\partial t} \Psi = \hat{h} \Psi = \left[-i\hbar c \,\hat{\sigma}_x \frac{\partial}{\partial x} + \left(kx + mc^2 \right) \hat{\sigma}_y \right] \Psi$$
(67)

(see Eqs. (9) and (10)). We have a particle inside the infinite region $\Omega = \mathbb{R}$, i.e., the entire real line. In general, the Hamiltonian operator in the Hilbert space $\mathscr{H} = \mathscr{L}^2(\Omega) \otimes \mathbb{C}^2$ is essentially self-adjoint on the set of infinitely differentiable two-component functions with compact support in Ω , no matter how great the potential \mathscr{S} is at infinity [19]. In fact, \hat{h} is a Hermitian operator on the same set of functions, i.e., it satisfies the hermiticity condition in Eq. (32) with $0 \to -\infty$ and $L \to +\infty$ (this is because all these functions vanish at infinity), but \hat{h} is self-adjoint on the Sobolev space $W^{1,2}(\mathbb{R})^2$. We denote the eigenfunctions of the Hamiltonian by $\psi_0, \psi_N^{(+)}$, and $\psi_N^{(-)}$, for the eigenvalues $E = 0, E = +\epsilon_N = +\sqrt{2\hbar c k N}$ (positive-energies), and $E = -\epsilon_N$ (negative-energies), with N = 1, 2, ..., respectively:

$$\hat{\mathbf{h}}\psi_0 = 0\psi_0 = 0, \quad \hat{\mathbf{h}}\psi_N^{(+)} = +\epsilon_N \ \psi_N^{(+)}, \quad \hat{\mathbf{h}}\psi_N^{(-)} = -\epsilon_N \ \psi_N^{(-)}.$$
(68)

Specifically, it was shown in Ref. [20] that the spectrum of \hat{h} is purely discrete and unbounded above and below. Explicitly, the eigenfunctions can be written as follows

$$\psi_0(x) = \psi_0 = a_0 \begin{bmatrix} 0\\1 \end{bmatrix} \exp\left(-\frac{k}{\hbar c} \frac{(x+x_0)^2}{2}\right),$$
(69)

and

$$\psi_{N}^{(+)}(x) = \psi_{N}^{(+)} = a_{N} \begin{bmatrix} -i\sqrt{2N} H_{N-1}\left(\sqrt{\frac{k}{\hbar c}}\left(x + x_{0}\right)\right) \\ H_{N}\left(\sqrt{\frac{k}{\hbar c}}\left(x + x_{0}\right)\right) \end{bmatrix} \exp\left(-\frac{k}{\hbar c}\frac{(x + x_{0})^{2}}{2}\right),$$
(70)

and

$$\psi_{N}^{(-)}(x) = \psi_{N}^{(-)} = \hat{\sigma}_{z}\psi_{N}^{(+)}$$

$$= a_{N} \begin{bmatrix} -i\sqrt{2N} H_{N-1}\left(\sqrt{\frac{k}{\hbar c}}\left(x+x_{0}\right)\right) \\ -H_{N}\left(\sqrt{\frac{k}{\hbar c}}\left(x+x_{0}\right)\right) \end{bmatrix} \exp\left(-\frac{k}{\hbar c}\frac{(x+x_{0})^{2}}{2}\right),$$
(71)

where N = 1, 2, ... In Eqs. (69)–(71), a_0 and a_N are (real) normalization constants, $x_0 \equiv mc^2/k$, and the functions H_N are Hermite polynomials. Additionally, notice that the complex eigenfunction $\psi_N^{(-)}$ is the same physical eigenstate as $(\psi_N^{(+)})^* = (\psi_N^{(+)})_C$, as expected. As usual, these eigenvectors satisfy the relations of orthogonality, namely,

$$\langle \psi_N^{(\pm)}, \psi_{N'}^{(\pm)} \rangle = 0, \text{ with } N \neq N', \quad \langle \psi_0, \psi_N^{(\pm)} \rangle = 0, \quad \langle \psi_N^{(+)}, \psi_{N'}^{(-)} \rangle = 0.$$
(72)

The most general Dirac wave function at t = 0 can be written as

$$\Psi\left(x,\,0\right) = \sum_{N=1}^{+\infty} c_N^{(+)}(0)\,\psi_N^{(+)}(x) + c_0(0)\,\psi_0(x) + \sum_{N=1}^{+\infty} c_N^{(-)}(0)\,\psi_N^{(-)}(x),\tag{73}$$

and assuming that the eigenfunctions are normalized, the condition $\|\Psi(x, 0)\| = 1$ leads to the restriction

$$\sum_{N=1}^{+\infty} |c_N^{(+)}(0)|^2 + |c_0(0)|^2 + \sum_{N=1}^{+\infty} |c_N^{(-)}(0)|^2 = 1,$$
(74)

where $c_N^{(\pm)}(0) \equiv c_N^{(\pm)}(t = 0)$ and $c_0(0) \equiv c_0(t = 0)$. The initial state for a Majorana particle is obtained by imposing the Majorana condition $(\Psi(x, 0) = \Psi_C(x, 0) = (\Psi(x, 0))^*)$ in Eq. (73), and this leads to the relations $c_N^{(-)}(0) = (c_N^{(+)}(0))^*$ and $c_0(0) = (c_0(0))^*$. Substituting the latter relations in Eq. (73), as well as that given before for the eigenfunctions,

we can write

$$\Psi\left(x, 0\right) = c_0(0)\psi_0(x) + \sum_{N=1}^{+\infty} [c_N^{(+)}(0)\psi_N^{(+)}(x) + \text{c. c.}],$$
with $\frac{1}{2} |c_0(0)|^2 + \sum_{N=1}^{+\infty} |c_N^{(+)}(0)|^2 = \frac{1}{2}.$
(75)

Once again, the real general solution $\Psi(x, t)$ is simply given by

$$\Psi\left(x,\,t\right) = c_0(0)\psi_0(x) + \sum_{N=1}^{+\infty} \left[c_N^{(+)}(0)\psi_N^{(+)}(x)\exp\left(-\mathrm{i}\frac{\epsilon_N}{\hbar}t\right) + \mathrm{c.\,c.} \right],\tag{76}$$

with the restriction that follows Eq. (75). Recently, for the problem addressed in this subsection, real particular solutions for a given *N* were obtained using the factorization method or the so-called supersymmetric procedure [21]. The gamma matrices used therein defined another Majorana representation, namely, $\hat{\gamma}^{0'} = -\hat{\sigma}_y$ and $\hat{\gamma}^{1'} = +i\hat{\sigma}_z$. These matrices and the matrices used in the present article ($\hat{\gamma}^0 = +\hat{\sigma}_y$ and $\hat{\gamma}^1 = -i\hat{\sigma}_z$) are related by $\hat{\gamma}^{\mu''} = \hat{\sigma}_x \hat{\gamma}^{\mu} \hat{\sigma}_x$ and $[\phi_1'' \phi_1'']^T = \hat{\sigma}_x [\phi_1 \phi_2]^T$.

Conclusions

A Majorana particle is its own antiparticle. Therefore, the condition that defines a particle of this type is given by $\Psi = \Psi_C$ (i.e., Ψ is invariant under charge conjugation). This is called the Majorana condition. This condition imposed on the Dirac wave function in the Majorana representation implies that this wave function must be real. On the other hand, in the Majorana representation, the gamma matrices must satisfy the relation $(i\hat{\gamma}^{\mu})^* = i\hat{\gamma}^{\mu}$, which implies that the Dirac operator $i\hat{\gamma}^{\mu}\partial_{\mu} - \frac{1}{\hbar c}(\mathscr{S} + mc^2)\hat{1}_2$ is real. Thus, in choosing the matrices in a Majorana representation, the solutions of the Dirac equation can be chosen to be real but do not need to be real. Thus, a Majorana particle cannot be described only with the Dirac equation in the Majorana representation without imposing the Majorana condition.

Precisely, by considering four examples or distinct physical situations (i.e., different borders and Lorentz scalar potentials), we constructed real general solutions of the Dirac equation in (1+1) dimensions, in the Majorana representation. The latter equation is a real system of coupled equations that can describe a Majorana particle as long as its solutions are real-valued. Clearly, the real general solutions obtained here are a superposition of real solutions for a given label or quantum number, but each of these real solutions is not a stationary solution but a superposition of positive and negative energy complex eigenstates (unless it is the real state corresponding to zero energy). Likewise, these real general solutions can perfectly be square integrable.

Because the equation describing the Majorana particle is the Dirac equation with a Lorentz scalar potential – although with the restriction imposed by the Majorana condition – the study of the supersymmetric procedure in these circumstances turns out to be more important. In particular, this task could give us new solutions that describe the Majorana particle. A very good reference on this matter is Ref. [22].

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