Introduction to String Compactification

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Summary. We present an elementary introduction to compactifications with unbroken supersymmetry. After explaining how this requirement leads to internal spaces of special holonomy we describe Calabi-Yau manifolds in detail. We also discuss orbifolds as examples of solvable string compactifications.

Contents

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1 Introduction

The need to study string compactification is a consequence of the fact that a quantum relativistic (super)string cannot propagate in any space-time background. The dynamics of a string propagating in a background geometry defined by the metric G_{MN} is governed by the *Polyakov action*

$$S_{\rm P} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}(X)\ . \tag{1.1}$$

Here σ^α, $\alpha=0,1$, are local coordinates on the string world-sheet Σ, $h_{\alpha\beta}$ is a metric on Σ with $h=\det h_{\alpha\beta}$, and X^M, $M=0,\dots,D-1$, are functions $\Sigma\hookrightarrow$ space-time $\mathcal{M}$ with metric $G_{MN}(X)$. α' is a constant of dimension (length)^2. $S_{\rm P}$ is the action of a two-dimensional non-linear σ-model with target space $\mathcal{M}$, coupled to two-dimensional gravity ($h_{\alpha\beta}$) where the D-dimensional metric G_{MN} appears as a coupling function (which generalizes the notion of a coupling constant). For flat space-time with metric $G_{MN}=\eta_{MN}$ the two-dimensional field theory is a free theory. The action (1.1) is invariant under local scale (Weyl) transformations $h_{\alpha\beta}\to e^{2\omega}h_{\alpha\beta}$, $X^M\to X^M$. One of the central principles of string theory is that when we quantize the two-dimensional field theory we must not loose this local scale invariance. In the path-integral quantization this means that it is not sufficient if the action is invariant because the integration measure might receive a non-trivial Jacobian which destroys the classical symmetry. Indeed, for the Polyakov action anomalies occur and produce a non-vanishing beta function $\beta^{(G)}_{MN}\equiv\alpha' R_{MN}+\mathcal{O}(\alpha'^2)$. Requiring $\beta^{(G)}_{MN}=0$ to maintain Weyl invariance gives the Einstein equations for the background metric: only their solutions are viable (perturbative) string backgrounds. But there are more restrictions.

Besides the metric, in the Polyakov action (1.1) other background fields can appear as coupling functions: an antisymmetric tensor-field $B_{MN}(X)$ and a scalar, the dilaton $\phi(X)$.[1] The background value of the dilaton determines the string coupling constant, i.e. the strength with which strings interact with each other. Taking into account the fermionic partners (under world-sheet supersymmetry) of X^M and $h_{\alpha\beta}$ gives beta functions for B_{MN} and ϕ that vanish for constant dilaton and zero antisymmetric field only if $D=10$. This defines the *critical dimension* of the supersymmetric string theories. We thus have to require that the background space-time $\mathcal{M}_{10}$ is a ten-dimensional Ricci-flat manifold with Lorentzian signature. Here we have ignored the $\mathcal{O}(\alpha'^2)$ corrections, to which we will briefly return later. The bosonic string which has critical dimension 26 is less interesting as it has no fermions in its excitation spectrum.

The idea of compactification arises naturally to resolve the discrepancy between the critical dimension $D=10$ and the number of observed dimensions $d=4$. Since $\mathcal{M}_{10}$ is dynamical, there can be solutions, consistent with the requirements imposed by local scale invariance on the world-sheet, which make the world *appear* four-dimensional. The simplest possibility is to have a background metric such that space-time takes the product form $\mathcal{M}_{10}=\mathcal{M}_4\times K_6$ where e.g. $\mathcal{M}_4$ is four-dimensional Minkowski space and K_6 is a compact space which admits a Ricci-flat metric. Moreover, to have escaped detection, K_6 must have dimensions of size smaller than the length scales already probed by particle accelerators. The type of theory observed in $\mathcal{M}_4$ will depend on properties of the compact space. For instance, in the classic analysis of superstring compactification of [2], it was found that when K_6 is a Calabi-Yau manifold, the resulting four-dimensional theory has a minimal number of supersymmetries [2]. One example of Calabi-Yau space discussed in [2] was the Z-manifold obtained by resolving the singularities of a $\mathrm{T}^6/\mathbb{Z}_3$ orbifold. It was soon noticed that string propagation on the singular orbifold was perfectly consistent and moreover exactly solvable [3]. These lectures provide an introduction to string compactifications on Calabi-Yau manifolds and orbifolds.

[1] There are other p-form fields, but their coupling to the world-sheet cannot be incorporated into the Polyakov action. The general statement is that the massless string states in the (NS,NS) sector, which are the metric, the anti-symmetric tensor and the dilaton, can be added to the Polyakov action. The massless p-forms in the (R,R) sector cannot. This can only be done within the so-called Green-Schwarz formalism and its extensions by Siegel and Berkovits; for review see [1].

The outline is as follows. In section 2 we give a short review of compactification à la Kaluza-Klein. Our aim is to explain how a particular choice of a compact manifold imprints itself on the four-dimensional theory. We also discuss how the requirement of minimal supersymmetry singles out Calabi-Yau manifolds. In section 3 we introduce some mathematical background: complex manifolds, Kähler manifolds, cohomology on complex manifolds. We then give a definition of Calabi-Yau manifolds and state Yau's theorem. Next we present the cohomology of Calabi-Yau manifolds and discuss their moduli spaces. As an application we work out the massless content of type II superstrings compactified on Calabi-Yau manifolds. In section 4 we study orbifolds, first explaining some basic properties needed to describe string compactification on such spaces. We systematically compute the spectrum of string states starting from the partition function. The techniques are next applied to compactify type II strings on $\mathrm{T}^{2n}/\mathbb{Z}_N$ orbifolds that are shown to allow unbroken supersymmetries. These toroidal Abelian orbifolds are in fact simple examples of spaces of special holonomy and the resulting lower-dimensional supersymmetric theories belong to the class obtained upon compactification on Calabi-Yau n-folds. We end with a quick look at recent progress. In Appendix A we fix our conventions and recall a few basic notions about spinors and Riemannian geometry. Two technical results which will be needed in the text are derived in Appendices B and C.

In these notes we review well known principles that have been applied in string theory for many years. There are several important developments which build on the material presented here which will not be discussed: mirror symmetry, D-branes and open strings, string dualities, compactification on manifolds with G_2 holonomy, etc. The lectures were intended for an audience of beginners in the field and we hope that they will be of use as preparation for advanced applications. We assume that the reader is already familiar with basic concepts in string theory that are well covered in textbooks [4, 5, 6]. But most of sections 2 and 3 do not use string theory at all. We have included many exercises whose solutions will eventually appear on [7].

2 Kaluza-Klein fundamentals

Kaluza and Klein unified gravity and electromagnetism in four dimensions by deriving both interactions from pure gravity in five dimensions. Generalizing this one might attempt to explain all known elementary particles and their interactions from a simple higher dimensional theory. String theory naturally lives in ten dimensions and so lends itself to the Kaluza-Klein program.

The discussion in this section is relevant for the field theory limit of string theory, where its massive excitation modes can be neglected. The dynamics of the massless modes is then described in terms of a low-energy effective action whose form is fixed by the requirement that it reproduces the scattering amplitudes as computed from string theory. However, when we compactify a string theory rather than a field theory, there are interesting additional features to which we return in section 4.

In the following we explain some basic results in Kaluza-Klein compactifications of field theories. For a comprehensive review see for instance [8] which cites the original literature. The basic material is well covered in [4] which also discusses the string theory aspects.

2.1 Dimensional reduction

Given a theory in D dimensions we want to derive the theory that results upon compactifying $D-d$ coordinates on an internal manifold K_{D-d}. As a simple example consider a real massless scalar in $D=5$ with action

$$S_0 = -\frac{1}{2}\int d^5x\, \partial_M\varphi\, \partial^M\varphi \ , \tag{2.1}$$

where $\partial^M = \eta^{MN}\partial_N$ with $\eta_{MN} = \eta^{MN} = \text{diag}\,(-,+,\cdots,+)$, $M,N = 0,\cdots,4$. The flat metric is consistent with the five-dimensional space $\mathcal{M}_5$ having product form $\mathcal{M}_5 = M_4 \times S^1$, where M_4 is four-dimensional Minkowski space and S^1 is a circle of radius R. We denote $x_M = (x_\mu, y)$, $\mu = 0,\cdots,3$, so that $y \in [0, 2\pi R]$. The field φ satisfies the equation of motion

$$\Box\varphi = 0 \quad \Rightarrow \quad \partial_\mu\partial^\mu\varphi + \partial_y^2\varphi = 0 \ . \tag{2.2}$$

Now, since $\varphi(x,y) = \varphi(x, y+2\pi R)$, we can write the Fourier expansion

$$\varphi(x,y) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{\infty} \varphi_n(x)e^{iny/R} \ . \tag{2.3}$$

Notice that $Y_n(y) \equiv \frac{1}{\sqrt{2\pi R}}e^{iny/R}$ are the orthonormalized eigenfunctions of ∂_y^2 on S^1. Substituting (2.3) in (2.2) gives

$$\partial_\mu\partial^\mu\varphi_n - \frac{n^2}{R^2}\varphi_n = 0 \ . \tag{2.4}$$

This clearly means that $\varphi_n(x)$ are 4-dimensional scalar fields with masses n/R. This can also be seen at the level of the action. Substituting (2.3) in (2.1) and integrating over y (using orthonormality of the Y_n) gives

$$S_0 = -\sum_{n=-\infty}^{\infty}\frac{1}{2}\int d^4x[\partial_\mu\varphi_n\,\partial^\mu\varphi_n^* + \frac{n^2}{R^2}\varphi_n^*\varphi_n] \ . \tag{2.5}$$

This again shows that in four dimensions there is one massless scalar φ_0 plus an infinite tower of massive scalars φ_n with masses n/R. We are usually interested in the limit $R \to 0$ in which only φ_0 remains light while the φ_n, $n \neq 0$, become very heavy and are discarded. We refer to this limit in which only the *zero mode* φ_0 is kept as *dimensional reduction* because we could obtain the same results demanding that $\varphi(x_M)$ be independent of y. More generally, dimensional reduction in this restricted sense is compactification on a torus $\mathrm{T}^{D\text{–}d}$, discarding massive modes, i.e. all states which carry momentum along the directions of the torus.

The important concept of zero modes generalizes to the case of curved internal compact spaces. However, it is only in the case of torus compactification that all zero modes are independent of the internal coordinates. This guarantees the consistency of the procedure of discarding the heavy modes in the sense that a solution of the lower-dimensional equations of motion is also a solution of the full higher-dimensional ones.

In D dimensions we can have other fields transforming in various representations of the Lorentz group $SO(1, D-1)$. We then need to consider how they decompose under the Lorentz group in the lower dimensions. Technically, we need to decompose the representations of $SO(1, D-1)$ under $SO(1, d-1) \times SO(D-d)$ associated to $M_d \times K_{D\text{–}d}$. For example, for a vector A_M transforming in the fundamental representation $\boldsymbol{D}$ we have the branching $\boldsymbol{D} = (\boldsymbol{d}, \boldsymbol{1}) + (\boldsymbol{1}, \boldsymbol{D\text{-}d})$. This just means that A_M splits into A_μ, $\mu = 0,\cdots,d\text{–}1$ and A_m, $m = d,\cdots,D-1$. A_μ is a vector under $SO(1, d\text{–}1)$ whereas A_m, for each m, is a singlet, i.e. the A_m appear as $(D-d)$ scalars in d dimensions. Similarly, a two-index antisymmetric tensor B_{MN} decomposes into $B_{\mu\nu}$, $B_{\mu m}$ and B_{mn}, i.e. into an antisymmetric tensor, vectors and scalars in d dimensions.

Exercise 2.1: Perform the dimensional reduction of:

- Maxwell electrodynamics.

$$S_1 = -\frac{1}{4}\int d^{4+n}x\, F_{MN}F^{MN} \quad , \quad F_{MN} = \partial_M A_N - \partial_N A_M \ . \tag{2.6}$$

- Action for a 2-form gauge field B_{MN}.

$$S_2 = -\frac{1}{12}\int d^{4+n}x\, H_{MNP}H^{MNP} \quad , \quad H_{MNP} = \partial_M B_{NP} + \text{cyclic} \ . \tag{2.7}$$

We also need to consider fields that transform as spinors under the Lorentz group. Here and below we will always assume that the manifolds considered are spin manifolds, so that spinor fields can be defined. As reviewed in Appendix A, in D dimensions, the Dirac matrices Γ^M are $2^{[D/2]} \times 2^{[D/2]}$-dimensional ($[D/2]$ denotes the integer part of $D/2$). The Γ^μ and Γ^m, used to build the generators of $SO(1,d-1)$ and $SO(D-d)$, respectively, then act on all $2^{[D/2]}$ spinor components. This means that an $SO(1,D-1)$ spinor transforms as a spinor under both $SO(1,d-1)$ and $SO(D-d)$. For example, a Majorana spinor ψ in $D=11$ decomposes under $SO(1,3) \times SO(7)$ as $\mathbf{32} = (\mathbf{4}, \mathbf{8})$, where $\mathbf{4}$ and $\mathbf{8}$ are respectively Majorana spinors of $SO(1,3)$ and $SO(7)$. Hence, dimensional reduction of ψ gives rise to eight Majorana spinors in $d = 4$.

We are mainly interested in compactification of supersymmetric theories that have a set of conserved spinorial charges Q^I, $I = 1, \cdots, \mathcal{N}$. Fields organize into supermultiplets containing both fermions and bosons that transform into each other by the action of the generators Q^I [9]. In each supermultiplet the numbers of on-shell bosonic and fermionic degrees of freedom do match and the masses of all fields are equal. Furthermore, the action that determines the dynamics of the fields is highly constrained by the requirement of invariance under supersymmetry transformations. For instance, for $D=11$, $\mathcal{N}=1$, there is a unique theory, namely eleven-dimensional supergravity. For $D = 10$, $\mathcal{N} = 2$ there are two different theories, non-chiral IIA (Q^1 and Q^2 are Majorana-Weyl spinors of opposite chirality) and chiral IIB supergravity (Q^1 and Q^2 of same chirality). For $D=10$, $\mathcal{N}=1$, a supergravity multiplet can be coupled to a non-Abelian super Yang-Mills multiplet provided that the gauge group is $E_8 \times E_8$ or $SO(32)$ to guarantee absence of quantum anomalies. The above theories describe the dynamics of M-theory and the various string theories at low energies.

One way to obtain four-dimensional supersymmetric theories is to start in $D = 11$ or $D = 10$ and perform dimensional reduction, i.e. compactify on a torus. For example, we have just explained that dimensional reduction of a $D=11$ Majorana spinor produces eight Majorana spinors in $d=4$. This means that starting with $D=11$, $\mathcal{N}=1$, in which Q is Majorana, gives a $d=4$, $\mathcal{N}=8$ theory upon dimensional reduction. As another interesting example, consider $D=10$, $\mathcal{N}=1$ in which Q is a Majorana-Weyl spinor. The ***16*** Weyl representation of $SO(1,9)$ decomposes under $SO(1,3) \times SO(6)$ as

$$\mathbf{16} = (\mathbf{2_L}, \bar{\mathbf{4}}) + (\mathbf{2_R}, \mathbf{4}) \ , \tag{2.8}$$

where $\mathbf{4}$, $\bar{\mathbf{4}}$ are Weyl spinors of $SO(6)$ and $\mathbf{2_{L,R}}$ are Weyl spinors of $SO(1,3)$ that are conjugate to each other. If we further impose the Majorana condition in $D = 10$, then dimensional reduction of Q gives rise to four Majorana spinors in $d=4$. Thus, $\mathcal{N}=1,2$ supersymmetric theories in $D=10$ yield $\mathcal{N}=4,8$ supersymmetric theories in $d=4$ upon dimensional reduction.

Toroidal compactification of superstrings gives theories with too many supersymmetries that are unrealistic because they are non-chiral, they cannot have the chiral gauge interactions observed in nature. Supersymmetric extensions of the Standard Model require $d=4$, $\mathcal{N}=1$. Such models have been extensively studied over the last 25 years (for a recent review, see [10]). One reason is that supersymmetry, even if it is broken at low energies, can explain why the mass of the Higgs boson does not receive large radiative corrections. Moreover, the additional particles and particular couplings required by supersymmetry lead to distinct experimental signatures that could be detected in future high energy experiments.

To obtain more interesting theories we must go beyond toroidal compactification. As a guiding principle we demand that some supersymmetry is preserved. As we will see, this allows a more precise characterization of the internal manifold. Supersymmetric string compactifications are moreover stable, in contrast to non-supersymmetric vacua that can be destabilized by tachyons or tadpoles. Now, we know that in the real world supersymmetry must be broken since otherwise the superpartner of e.g. the electron would have been observed. Supersymmetry breaking in string theory is still an open problem.

2.2 Compactification, supersymmetry and Calabi-Yau manifolds

Up to now we have not included gravity. When a metric field G_{MN} is present, the fact that space-time $\mathcal{M}_D$ has a product form $\mathcal{M}_d \times K_{D-d}$, with K_{D-d} compact, must follow from the dynamics. If the equations of motion have such a solution, we say that the system admits *spontaneous compactification.* The vacuum expectation value (vev) of G_{MN} then satisfies

$$\langle G_{MN}(x,y)\rangle = \begin{pmatrix} \bar{g}_{\mu\nu}(x) & 0 \\ 0 & \bar{g}_{mn}(y) \end{pmatrix} , \tag{2.9}$$

where x_μ and y_m are the coordinates of $\mathcal{M}_d$ and K_{D-d} respectively. Note that with this Ansatz there are no non-zero components of the Christoffel symbols and the Riemann tensor which carry both Latin and Greek indices. An interesting generalization of (2.9) is to keep the product form but with the metric components on $\mathcal{M}_d$ replaced by $e^{2A(y)}\bar{g}_{\mu\nu}(x)$, where $A(y)$ is a so-called 'warp factor' [11]. This still allows maximal space-time symmetry in $\mathcal{M}_d$. For instance $\langle G_{\mu\nu}(x,y)\rangle = e^{2A(y)}\eta_{\mu\nu}$ is compatible with d-dimensional Poincaré symmetry. In these notes we do not consider such warped product metrics.

We are mostly interested in D-dimensional supergravity theories and we will search for compactifications that preserve some degree of supersymmetry. Instead of analyzing whether the equations of motion, which are highly nonlinear, admit solutions of the form (2.9), it is then more convenient to demand (2.9) and require unbroken supersymmetries in $\mathcal{M}_d$. A posteriori it can be checked that the vevs obtained for all fields are compatible with the equations of motion.

We thus require that the vacuum satisfies $\bar{\epsilon}Q|0\rangle = 0$ where $\epsilon(x^M)$ parametrizes the supersymmetry transformation which is generated by Q, both Q and ϵ being spinors of $SO(1,D{-}1)$. This, together with $\delta_\epsilon\Phi = [\bar{\epsilon}Q, \Phi]$, means that $\langle\delta_\epsilon\Phi\rangle \equiv \langle 0|[\bar{\epsilon}Q,\Phi]|0\rangle = 0$ for every field generically denoted by Φ. Below we will be interested in the case where $\mathcal{M}_d$ is Minkowski space. Then, with the exception of a vev for the metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and a d-form $\bar{F}_{\mu_1\ldots\mu_d} = \epsilon_{\mu_1\ldots\mu_d}$, a non-zero background value of any field which is not a $SO(1,d-1)$ scalar, would reduce the symmetries of Minkowski space. In particular, since fermionic fields are spinors that transform non-trivially under $SO(1,d{-}1)$, $\langle\Phi_{\rm Fermi}\rangle = 0$. Hence, $\langle\delta_\epsilon\Phi_{\rm Bose}\rangle \sim \langle\Phi_{\rm Fermi}\rangle = 0$ and we only need to worry about $\langle\delta_\epsilon\Phi_{\rm Fermi}\rangle$. Now, among the $\Phi_{\rm Fermi}$ in supergravity there is always the gravitino ψ_M (or $\mathcal{N}$ gravitini if there are $\mathcal{N}$ supersymmetries in higher dimensions) that transforms as

$$\delta_\epsilon\psi_M = \nabla_M\,\epsilon + \cdots\ , \tag{2.10}$$

where ∇_M is the covariant derivative defined in Appendix A. The ... stand for terms which contain other bosonic fields (dilaton, B_{MN} and p-form fields) whose vevs are taken to be zero. Then, $\langle\delta_\epsilon\psi_M\rangle = 0$ gives

$$\langle\nabla_M\epsilon\rangle \equiv \bar{\nabla}_M\epsilon = 0 \qquad \Rightarrow \quad \bar{\nabla}_m\epsilon = 0 \quad \text{and} \quad \bar{\nabla}_\mu\epsilon = 0\ . \tag{2.11}$$

Notice that in $\bar{\nabla}_M$ there appears the vev of the spin connection $\bar{\omega}$. Spinor fields ϵ, which satisfy (2.11) are covariantly constant (in the vev metric); they are also called *Killing spinors.*

The existence of Killing spinors, which is a necessary requirement for a supersymmetric compactification, restricts the class of manifolds on which we may compactify. To see this explicitly, we iterate (2.11) to obtain the integrability condition (since the manipulations until (2.14) are completely general, we drop the bar)

$$[\nabla_M,\nabla_N]\epsilon = \frac{1}{4}R_{MN}{}^{AB}\Gamma_{AB}\epsilon = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\epsilon = 0\ , \tag{2.12}$$

where $\Gamma_{AB} = \frac{1}{2}[\Gamma_A,\Gamma_B]$ and R_{MNPQ} is the Riemann tensor in the D-dimensional background.

Exercise 2.2: Verify (2.12) using (A.12).

Next we multiply by Γ^N and use the Γ property

$$\Gamma^N\Gamma^{PQ} = \Gamma^{NPQ} + G^{NP}\Gamma^Q - G^{NQ}\Gamma^P\ , \tag{2.13}$$

where Γ^{NPQ} is defined in (A.2). The Bianchi identity

$$R_{MNPQ} + R_{MQNP} + R_{MPQN} = 0 \tag{2.14}$$

implies that $\Gamma^{NPQ} R_{MNPQ} = 0$. In this way we arrive at

$$\bar{R}_{MQ} \bar{\Gamma}^{Q} \epsilon = 0 \ . \tag{2.15}$$

With $\det \Gamma \neq 0$ it follows that a necessary condition for the existence of a Killing spinor on a Riemannian manifold is the vanishing of its Ricci tensor:

$$\bar{R}_{MQ} = 0 \ . \tag{2.16}$$

This is the same condition as that obtained from the requirement of Weyl invariance at the level of the string world-sheet and it is also the equation of motion derived from the supergravity action if all fields except the metric are set to zero. It already followed from the Ansatz $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ that $\bar{R}_{\mu\nu} = 0$ and that ϵ does not depend on the coordinates x^μ. Equation (2.16) also gives

$$\bar{R}_{mn} = 0 \ . \tag{2.17}$$

Hence, the internal $K_{D\text{–}d}$ is a compact Ricci-flat manifold.

One allowed solution is $K_{D\text{–}d} = \mathrm{T}^{D\text{–}d}$, i.e. a $(D-d)$ torus that is compact and flat. This means that dimensional reduction is always possible and, since ϵ is constant because in this case $\bar{\nabla}_m \epsilon = \partial_m \epsilon = 0$, it gives the maximum number of supersymmetries in the lower dimensions. The fact that supersymmetry requires $K_{D\text{–}d}$ to be Ricci-flat is a very powerful result. For example, it is known that Ricci-flat compact manifolds do not admit Killing vectors other than those associated with tori. Equivalently, the Betti number b_1 only gets contributions from non-trivial cycles associated to tori factors in $K_{D\text{–}d}$. The fact that the internal manifold must have Killing spinors encodes much more information. To analyze this in more detail below we specialize to a six-dimensional internal K_6 which is the case of interest for string compactifications from $D=10$ to $d=4$.

Before doing this we need to introduce the concept of holonomy group $\mathcal{H}$ [12, 13]. Upon parallel transport along a closed curve on an m-dimensional manifold, a vector v is rotated into Uv. The set of matrices obtained in this way forms $\mathcal{H}$. The U's are necessarily matrices in $O(m)$ which is the tangent group of the Riemannian K_m. Hence $\mathcal{H} \subseteq O(m)$. For manifolds with an orientation the stronger condition $\mathcal{H} \subseteq SO(m)$ holds. Now, from (A.14) it follows that for a simply-connected manifold to have non-trivial holonomy it has to have curvature. Indeed, the Riemann tensor (and its covariant derivatives), when viewed as a Lie-algebra valued two-form, generate $\mathcal{H}$. If the manifold is not simply connected, the Riemann tensor and its covariant derivatives only generate the identity component of the holonomy group, called the *restricted holonomy group* $\mathcal{H}_0$ for which $\mathcal{H}_0 \subseteq SO(m)$. Non-simply connected manifolds can have non-trivial $\mathcal{H}$ without curvature, as exemplified in the following exercise.

Exercise 2.3: Consider the manifold $S^1 \otimes \mathbb{R}^n$ endowed with the metric

$$ds^2 = R^2 d\theta^2 + (dx^i + \Omega^i{}_j x^j d\theta)^2 \ , \tag{2.18}$$

where $\Omega^i{}_j$ is a constant anti-symmetric matrix, i.e. a generator of the rotation group $SO(n)$ and R is the radius of S^1. Show that this metric has vanishing curvature but that nevertheless a vector, when parallel transported around the circle, is rotated by an element of $SO(n)$.

Under parallel transport along a loop in K_6, spinors are also rotated by elements of $\mathcal{H}$. But a covariantly constant spinor such as ϵ remains unchanged. This means that ϵ is a singlet under $\mathcal{H}$. But ϵ is an $SO(6)$ spinor and hence it has right- and left-chirality pieces that transform respectively as $\mathbf{4}$ and $\bar{\mathbf{4}}$ of

$SO(6) \simeq SU(4)$. How can ϵ be an $\mathcal{H}$-singlet ? Suppose that $\mathcal{H} = SU(3)$. Under $SU(3)$ the $\mathbf{4}$ decomposes into a triplet and a singlet: $\mathbf{4}_{SU(4)} = (\mathbf{3}+\mathbf{1})_{SU(3)}$. Thus, if $\mathcal{H} = SU(3)$ there is one covariantly constant spinor of positive and one of negative chirality, which we denote $\epsilon_\pm$. If $\mathcal{H}$ were $SU(2)$ there would be two right-handed and two left-handed covariantly constant spinors since under $SU(2)$ the $\mathbf{4}$ decomposes into a doublet and two singlets. There could be as many as four covariantly constant spinors of each chirality as occurs when $K_6 = \mathrm{T}^6$ and $\mathcal{H}_0$ is trivial since the torus is flat.

Let us now pause to show that if K_6 has $SU(3)$ holonomy, the resulting theory in $d{=}4$ has precisely $\mathcal{N}{=}1$ supersymmetry if it had $\mathcal{N}{=}1$ in $D{=}10$. Taking into account the decomposition (2.8) and the discussion in the previous paragraph, we see that the allowed supersymmetry parameter takes the form

$$\epsilon = \epsilon_R \otimes \epsilon_+ + \epsilon_L \otimes \epsilon_- \ . \tag{2.19}$$

Since ϵ is also Majorana it must be that $\epsilon_R = \epsilon_L^*$ and hence ϵ_R and ϵ_L form just a single Majorana spinor, associated to a single supersymmetry generator. Similarly, if K_6 has $SU(2)$ holonomy the resulting $d{=}4$ theory will have $\mathcal{N}{=}2$ supersymmetry. Obviously, the number of supersymmetries in $d{=}4$ is doubled if we start from $\mathcal{N}{=}2$ in $D{=}10$.

$2n$-dimensional compact Riemannian manifolds with $SU(n) \subset SO(2n)$ holonomy are *Calabi-Yau manifolds* CY_n. We have just seen that they admit covariantly constant spinors and that they are Ricci-flat. We will learn much more about Calabi-Yau manifolds in the course of these lectures and we will also make the definition more precise. For $n = 1$ there is only one CY_1, namely the torus T^2. The only CY_2 is the K3 manifold. For $n \geq 3$ there is a huge number. We will give simple examples of CY_3 in section 3. Many more can be found in [14]. We want to remark that except for the trivial case $n = 1$ no metric with $SU(n)$ holonomy on any CY_n is known explicitly. Existence and uniqueness have, however, been shown (cf. section 3).

Calabi-Yau manifolds are a class of *manifolds with special holonomy.* Generically on an oriented manifold one has $\mathcal{H} \simeq SO(m)$. Then the following question arises: which subgroups $G \subset SO(m)$ do occur as holonomy groups of Riemannian manifolds ? For the case of simply connected manifolds which are neither symmetric nor locally a product of lower dimensional manifolds, this question was answered by Berger. His classification along with many of the properties of the manifolds is discussed at length in [12, 13]. All types of manifolds with special holonomy do occur in the context of string compactification, either as the manifold on which we compactify or as moduli spaces (cf. section 3.6).

Exercise 2.4: Use simple group theory to work out the condition on the holonomy group of seven- and eight-dimensional manifolds which gives the minimal amount of supersymmetry if one compactifies eleven-dimensional supergravity to four or three dimensions or ten-dimensional supergravity to $d = 3$ and $d = 2$, respectively.

Going back to the important case, $\mathcal{N}{=}1$, $D{=}10$, $d{=}4$, and the requirement of unbroken supersymmetry we find the following possibilities. The internal K_6 can be a torus T^6 with trivial holonomy and hence ϵ leads to $d{=}4$, $\mathcal{N}{=}4$ supersymmetry. K_6 can also be a product $\mathrm{K3} \times \mathrm{T}^2$ with $SU(2)$ holonomy and ϵ leads to $\mathcal{N}{=}2$ in $d{=}4$. Finally, K_6 can be a CY_3 that has $SU(3)$ holonomy so that ϵ gives $d{=}4$, $\mathcal{N}{=}1$ supersymmetry. These are the results for heterotic and type I strings. For type II strings the number of supersymmetries in the lower dimensions is doubled since we start from $\mathcal{N}{=}2$ in $D{=}10$.

Let us also consider compactifications from $\mathcal{N} = 1$, $D = 10$ to $d = 6$. In this case unbroken supersymmetry requires K_4 to be the flat torus T^4 or the K3 manifold with $SU(2)$ holonomy. Toroidal compactification does not reduce the number of real supercharges (16 in $\mathcal{N}{=}1$, $D{=}10$), thus when the internal manifold is T^4 the theory in $d{=}6$ has $\mathcal{N}{=}2$, or rather (1,1), supersymmetry. Here the notation indicates that one supercharge is a left-handed and the other a right-handed Weyl spinor. The $SO(1,5)$ Weyl spinors are complex since a Majorana-Weyl condition cannot be imposed in $d{=}6$. Compactification on K3 gives $d\!=\!6$, $\mathcal{N}\!=\!1$, or rather (1,0), supersymmetry. This can be understood from the decomposition of the **16** Weyl representation of $SO(1,9)$ under $SO(1,5)\times SO(4)$,

$$\mathbf{16} = (\mathbf{4_L},\mathbf{2}) + (\mathbf{4_R},\mathbf{2'}) \ , \tag{2.20}$$

where $\mathbf{4_{L,R}}$ and $\mathbf{2}$, $\mathbf{2'}$ are Weyl spinors of $SO(1,5)$ and $SO(4)$. In both groups each Weyl representation is its own conjugate. Since the supersymmetry parameter ϵ in $D\!=\!10$ is Majorana-Weyl, its $(\mathbf{4_L},\mathbf{2})$ piece has only eight real components which form only one complex $\mathbf{4_L}$ and likewise for $(\mathbf{4_R},\mathbf{2'})$. Then, if the holonomy is trivial, ϵ gives one $\mathbf{4_L}$ plus one $\mathbf{4_R}$ supersymmetry in $d=6$. Instead, if the holonomy is $SU(2)\subset SO(4)\simeq SU(2)\times SU(2)$, only one $SO(4)$ spinor, say $\mathbf{2}$, is covariantly constant and then ϵ gives only one $\mathbf{4_L}$ supersymmetry. Starting from $\mathcal{N}=2$ in $D\!=\!10$ there are the following possibilities. Compactification on T^4 gives (2,2) supersymmetry for both the non-chiral IIA and the chiral IIB superstrings. However, compactification on K3 gives (1,1) supersymmetry for IIA but (2,0) supersymmetry for IIB.

From the number of unbroken supersymmetries in the lower dimensions we can already observe hints of string dualities, i.e. equivalences of the compactifications of various string theories. For example, in $d\!=\!6$, the type IIA string on K3 is dual to the heterotic string on T^4 and in $d\!=\!4$, type IIA on CY_3 is dual to heterotic on K3 $\times$ T^2. On the heterotic side non-Abelian gauge groups are perturbative but on the type IIA side they arise from non-perturbative effects, namely D-branes wrapping homology cycles inside the K3 surface. We will not discuss string dualities in these lectures. For a pedagogical introduction, see [6].

2.3 Zero modes

We now wish to discuss Kaluza-Klein reduction when compactifying on curved internal spaces. Our aim is to determine the resulting theory in d dimensions. To begin we expand all D-dimensional fields, generically denoted $\Phi^{mn\cdots}_{\mu\nu\cdots}(x,y)$, around their vacuum expectation values

$$\Phi^{mn\cdots}_{\mu\nu\cdots}(x,y) = \langle\Phi^{mn\cdots}_{\mu\nu\cdots}(x,y)\rangle + \varphi^{mn\cdots}_{\mu\nu\cdots}(x,y) \ . \tag{2.21}$$

We next substitute in the D-dimensional equations of motion and use the splitting (2.9) of the metric. Keeping only linear terms, and possibly fixing gauge, gives generic equations

$$\mathcal{O}_d\varphi^{mn\cdots}_{\mu\nu\cdots} + \mathcal{O}_{\rm int}\varphi^{mn\cdots}_{\mu\nu\cdots} = 0 \ , \tag{2.22}$$

where $\mathcal{O}_d$, $\mathcal{O}_{\rm int}$ are differential operators of order p ($p=2$ for bosons and $p=1$ for fermions) that depend on the specific field.

We next expand $\varphi^{mn\cdots}_{\mu\nu\cdots}$ in terms of eigenfunctions $Y_a^{mn\cdots}(y)$ of $\mathcal{O}_{\rm int}$ in $K_{D\text{–}d}$. This is

$$\varphi^{mn\cdots}_{\mu\nu\cdots}(x,y) = \sum_a \varphi_{a\mu\nu\cdots}(x)\ Y_a^{mn\cdots}(y) \ . \tag{2.23}$$

Since $\mathcal{O}_{\rm int}Y_a^{mn\cdots}(y) = \lambda_a Y_a^{mn\cdots}(y)$, from (2.22) we see that the eigenvalues λ_a determine the masses of the d-dimensional fields $\varphi_{a\mu\nu\cdots}(x)$. With R a typical dimension of $K_{D\text{–}d}$, $\lambda_a\sim 1/R^p$. We again find that in the limit $R\to 0$ only the zero modes of $\mathcal{O}_{\rm int}$ correspond to massless fields $\varphi_{0\mu\nu\cdots}(x)$.

To obtain the effective d-dimensional action for the massless fields φ_0 in general it is not consistent to simply set the massive fields, i.e. the coefficients of the higher harmonics, to zero [8]. The problem with such a truncation is that the heavy fields, denoted φ_h, might induce interactions of the φ_0 that are not suppressed by inverse powers of the heavy mass. This occurs for instance when there are cubic couplings $\varphi_0\,\varphi_0\,\varphi_h$. When the zero modes $Y_0(y)$ are constant or covariantly constant a product of them is also a zero mode and then by orthogonality of the $Y_a(y)$ terms linear in φ_h cannot appear after integrating over the extra dimensions, otherwise they might be present and generate corrections to quartic and higher order couplings of the φ_0. Even when the heavy fields cannot be discarded it might be possible to consistently determine the effective action for the massless fields [15].

We have already seen that for scalar fields $\mathcal{O}_{\rm int}$ is the Laplacian Δ. On a compact manifold Δ has only one scalar zero mode, namely a constant and hence a scalar in D dimensions produces just one massless scalar in d dimensions. An important and interesting case is that of Dirac fields in which both $\mathcal{O}_d$ and $\mathcal{O}_{\rm int}$ are Dirac operators $\Gamma \cdot \nabla$. The number of zero modes of $\not{\nabla} \equiv \Gamma^m \nabla_m$ happen to depend only on topological properties of the internal manifold $K_{D\text{–}d}$ and can be determined using index theorems [4]. When the internal manifold is Calabi-Yau we can also exploit the existence of covariantly constant spinors. For instance, from the formula $\not{\nabla}^2 = \nabla^m\nabla_m$, which is valid on a Ricci-flat manifold, it follows that when K_6 is a CY_3, the Dirac operator has only two zero modes, namely the covariantly constant ϵ_+ and ϵ_-.

Among the massless higher dimensional fields there are usually p-form gauge fields $A^{(p)}$ with field strength $F^{(p+1)} = dA^{(p)}$ and action

$$S_p = -\frac{1}{2(p+1)!}\int_{\mathcal{M}_D} F^{(p+1)} \wedge *F^{(p+1)} \ . \tag{2.24}$$

After fixing the gauge freedom $A^{(p)} \to A^{(p)} + d\Lambda^{(p-1)}$ by imposing $d^*A^{(p)} = 0$, the equations of motion are

$$\Delta_D A^{(p)} = 0\,, \qquad \Delta_D = dd^* + d^*d \ . \tag{2.25}$$

If the metric splits into a d-dimensional and a $(D-d)$-dimensional part, as in (2.9), the Laplacian Δ_D also splits $\Delta_D = \Delta_d + \Delta_{D-d}$. Then, $\mathcal{O}_{\rm int}$ is the Laplacian Δ_{D-d}. The number of massless d-dimensional fields is thus given by the number of zero modes of the internal Laplacian. This is a cohomology problem, as we will see in detail in section 3. In particular, the numbers of zero modes are given by Betti numbers b_r. For example, there is a 2-form that decomposes $B_{MN} \to B_{\mu\nu} \oplus B_{\mu m} \oplus B_{mn}$. Each term is an n-form with respect to the internal manifold, where n is easily read from the decomposition. Thus, from $B_{\mu\nu}$ we obtain only one zero mode since $b_0 = 1$, from $B_{\mu m}$ we obtain b_1 modes that are vectors in d dimensions and from B_{mn} we obtain b_2 modes that are scalars in d dimensions. In general, from a p-form in D dimensions we obtain b_n massless fields, $n = 0, \cdots, p$, that correspond to $(p-n)$-forms in d dimensions.

Let us now consider zero modes of the metric that decomposes $g_{MN} \to g_{\mu\nu} \oplus g_{\mu m} \oplus g_{mn}$. From $g_{\mu\nu}$ there is only one zero mode, namely the lower dimensional graviton. Massless modes coming from $g_{\mu m}$, that would behave as gauge bosons in d dimensions, can appear only when $b_1 \neq 0$ and the internal manifold has continuous isometries. Massless modes arising from g_{mn} correspond to scalars in d dimensions. To analyze these modes we write $g_{mn} = \bar{g}_{mn} + h_{mn}$. We know that a necessary condition for the fluctuations h_{mn} not to break supersymmetry is $R_{mn}(\bar{g} + h) = 0$ just as $R_{mn}(\bar{g}) = 0$. Thus, the h_{mn} are degeneracies of the vacuum, they preserve the Ricci-flatness.

The h_{mn} are usually called moduli. They are free parameters in the compactification which change the size and shape of the manifold but not its topology. For instance, a circle S^1 has one modulus, namely its radius R. The fact that any value of R is allowed manifests itself in the space-time theory as a massless scalar field with vanishing potential. The 2-torus, that has one Kähler modulus and one complex structure modulus, is another instructive example. To explain its moduli we define T^2 by identifications in a lattice Λ. This means $\mathrm{T}^2 = \mathbb{R}^2/\Lambda$. We denote the lattice vectors e_1, e_2 and define a metric $G_{mn} = \mathrm{e}_m \cdot \mathrm{e}_n$. The Kähler modulus is just the area $\sqrt{\det G}$. If there is an antisymmetric field B_{mn} then it is natural to introduce the complex Kähler modulus T via

$$T = \sqrt{\det G} + iB_{12} \ . \tag{2.26}$$

The complex structure modulus, denoted U, is

$$U = -i\frac{|\mathrm{e}_2|}{|\mathrm{e}_1|}e^{i\varphi(\mathrm{e}_1,\mathrm{e}_2)} = \frac{1}{G_{11}}(\sqrt{\det G} - iG_{12}) \ . \tag{2.27}$$

U is related to the usual modular parameter by $\tau = iU$. τ can be written as a ratio of periods of the holomorphic 1-form $\Omega = dz$. Specifically, $\tau = \int_{\gamma_2} dz / \int_{\gamma_1} dz$, where γ_1, γ_2 are the two non-trivial one-cycles (associated to $\mathrm{e}_1, \mathrm{e}_2$). While all tori are diffeomorphic as real manifolds, there is no holomorphic map between two tori with complex structures τ and τ' unless they are related by a $SL(2,\mathbb{Z})$ modular transformation, cf. (4.28). This is a consequence of the geometric freedom to make integral changes of lattice basis, as long as the volume of the unit cell does not change (see e.g. [16]). Furthermore, in string theory compactification there is a T-duality symmetry, absent in field theory, that in circle compactification identifies R and $R' = \alpha'/R$, whereas in T^2 compactification identifies all values of T related by an $SL(2,\mathbb{Z})_T$ transformation (for review, see e.g. [17]). Compactification on a torus will thus lead to two massless fields, also denoted U and T, with completely arbitrary vevs but whose couplings to other fields are restricted by invariance of the low-energy effective action under $SL(2,\mathbb{Z})_U$ and $SL(2,\mathbb{Z})_T$.

The metric moduli of CY 3-folds are also divided into Kähler moduli and complex structure moduli. This will be explained in section 3.6.

Our discussion of compactification so far has been almost entirely in terms of field theory, rather than string theory. Of course, what we have learned about compactification is also relevant for string theory, since at low energies, where the excitation of massive string modes can be ignored, the dynamics of the massless modes is described by a supergravity theory in ten dimensions (for type II strings) coupled to supersymmetric Yang-Mills theory (for type I and heterotic strings).

But there are striking differences between compactifications of field theories and string theories. When dealing with strings, it is not the classical geometry (or even topology) of the space-time manifold $\mathcal{M}$ which is relevant. One dimensional objects, such as strings, probe $\mathcal{M}$ differently from point particles. Much of the attraction of string theory relies on the hope that the modification of the concept of classical geometry to 'string geometry' at distances smaller than the string scale $l_s = \sqrt{\alpha'}$ (which is of the order of the Planck length[2], i.e. $\sim 10^{-33}$cm) will lead to interesting effects and eventually to an understanding of physics in this distance range. At distances large compared to l_s a description in terms of point particles should be valid and one should recover classical geometry.

One particular property of string compactification as compared to point particles is that there might be more than one manifold K_m which leads to identical theories. This resembles the situation of point particles on so-called isospectral manifolds. However, in string theory the invariance is more fundamental, as no experiment can be performed to distinguish between the manifolds. This is an example of a duality, of which many are known. T-duality of the torus compactification is one simple example which was already mentioned. A particularly interesting example which arose from the study of Calabi-Yau compactifications is *mirror symmetry* . It states that for any Calabi-Yau manifold X there exists a mirror manifold $\hat{X}$, such that $\mathrm{IIA}(X) = \mathrm{IIB}(\hat{X})$. Here the notation $\mathrm{IIA}(X)$ means the full type IIA string theory, including all perturbative and non-perturbative effects, compactified on X. For the heterotic string with standard embedding of the spin connection in the gauge connection [2] mirror symmetry means $\mathrm{het}(X) = \mathrm{het}(\hat{X})$. The manifolds comprising a mirror pair are very different, e.g. in terms of their Euler numbers $\chi(X) = -\chi(\hat{X})$. The two-dimensional torus, which we discussed above, is its own mirror manifold, but mirror symmetry exchanges the two types of moduli: $U \leftrightarrow T$. In compactifications on Calabi-Yau 3-folds, mirror symmetry also exchanges complex structure and Kähler moduli between X and $\hat{X}$.

Mirror symmetry in string compactification is a rather trivial consequence of its formulation in the language of two-dimensional conformal field theory. However, when cast in the geometric language, it becomes highly non-trivial and has lead to surprising predictions in algebraic geometry. Except for a

[2]This is fixed by the identification of one of the massless excitation modes of the closed string with the graviton and comparing its self-interactions, as computed from string theory, with general relativity. This leads to a relation between Newton's constant and α'.

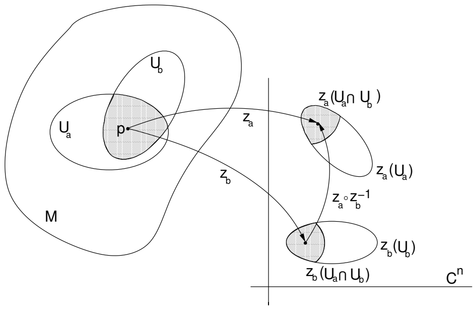


Figure 1: Coordinate maps on complex manifolds

few additional comments at the end of section 3.6 we will not discuss mirror symmetry in these lectures. An up-to-date extensive coverage of most mathematical and physical aspects of mirror symmetry has recently appeared [18].

3 Complex manifolds, Kähler manifolds, Calabi-Yau manifolds

3.1 Complex manifolds

In the previous chapter we have seen how string compactifications which preserve supersymmetry directly lead to manifolds with $SU(3)$ holonomy. These manifolds have very special properties which we will discuss in this chapter. In particular they can be shown to be complex manifolds. We begin this chapter with a review of complex manifolds and of some of the mathematics necessary for the discussion of CY manifolds. Throughout we assume some familiarity with real manifolds and Riemannian geometry. None of the results collected in this chapter are new, but some of the details we present are not readily available in the (physics) literature. Useful references are [19, 4, 20, 21, 22] (physics), [23, 24, 25, 26, 12, 27] (mathematics) and, in particular, [28]. In this section we use Greek indices for the (real) coordinates on the compactification manifold, which we will generically call M.

A *complex manifold* M is a differentiable manifold admitting an open cover $\{U_a\}_{a\in A}$ and coordinate maps $z_a : U_a \to \mathbb{C}^n$ such that $z_a \circ z_b^{-1}$ is holomorphic on $z_b(U_a \cap U_b) \subset \mathbb{C}^n$ for all a, b. $z_a = (z_a^1, \ldots, z_a^n)$ are *local holomorphic coordinates* and on overlaps $U_a \cap U_b$, $z_a^i = f_{ab}^i(z_b)$ are holomorphic functions, i.e. they do not depend on $\bar{z}_b^i$. (When considering local coordinates we will often drop the subscript which refers to a particular patch.) A complex manifold thus looks locally like $\mathbb{C}^n$. Transition functions from one coordinate patch to another are holomorphic functions. An *atlas* $\{U_a, z_a\}_{a\in A}$ with the above properties defines a *complex structure* on M. If the union of two such atlases has again the same properties, they are said to define the same complex structure; cf. differential structure in the real case, which is defined by (equivalence classes) of C^∞ atlases. n is called the *complex dimension* of M: $n = \dim_{\mathbb{C}}(M)$. Clearly, a complex manifold can be viewed as a real manifold with even (real) dimension, i.e. $m = 2n$. Not all real manifolds of even dimension can be endowed with a complex structure. For instance, among the even-dimensional spheres S^{2n}, only S^2 admits a complex structure. However, direct products of odd-dimensional spheres always admit a complex structure ([24], p.4).

Example 3.1: $\mathbb{C}^n$ is a complex manifold which requires only one single coordinate patch. We can consider $\mathbb{C}^n$ as a real manifold if we identify it with $\mathbb{R}^{2n}$ in the usual way by decomposing the complex coordinates into their real and imaginary parts ($i = \sqrt{-1}$):

$$z^j = x^j + iy^j\,, \quad \bar{z}^j = x^j - iy^j\,, \quad j = 1\ldots, n\,. \tag{3.1}$$

We will sometimes use the notation $x^{n+j} \equiv y^j$. For later use we give the decomposition of the partial derivatives

$$\partial_j \equiv \frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \bar{\partial}_j \equiv \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right). \tag{3.2}$$

and the differentials

$$dz^j = dx^j + idy^j\,, \qquad d\bar{z}^j = dx^j - idy^j\,. \tag{3.3}$$

Locally, on any complex manifold, we can always choose real coordinates as the real and imaginary parts of the holomorphic coordinates. A complex manifold is thus also a real analytic manifold. Moreover, since $\det \frac{\partial(x_a^1,\ldots,x_a^{2n})}{\partial(x_b^1,\ldots,x_b^{2n})} = \left|\det \frac{\partial(z_a^i,\ldots,z_a^n)}{\partial(z_b^1,\ldots,\partial z_b^n)}\right|^2 > 0$ on $U_a \cap U_b$, any complex manifold is orientable.

Example 3.2: A very important example, for reasons we will learn momentarily, is n-dimensional *complex projective space* $\mathbb{CP}^n$, or, simply, $\mathbb{P}^n$. $\mathbb{P}^n$ is defined as the set of (complex) lines through the origin of $\mathbb{C}^{n+1}$. A line through the origin can be specified by a single point and two points z and w define the same line iff there exists $\lambda \in \mathbb{C}^* \equiv \mathbb{C} - \{0\}$ such that $z = (z^0, z^1 \ldots, z^n) = (\lambda w^0, \lambda w^1, \ldots, \lambda w^n) \equiv \lambda \cdot w$. We thus have

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*} \tag{3.4}$$

The coordinates $z^0, \ldots, z^n$ are called *homogeneous coordinates* on $\mathbb{P}^n$. Often we write $[z] = [z^0 : z^1 : \ldots : z^n]$. $\mathbb{P}^n$ can be covered by $n+1$ coordinate patches $U_i = \{[z] : z^i \neq 0\}$, i.e. U_i consists of those lines through the origin which do not lie in the hyperplane $z^i = 0$. (Hyperplanes in $\mathbb{P}^n$ are $n-1$-dimensional submanifolds, or, more generally, codimension-one submanifolds.) In U_i we can choose local coordinates as $\xi_i^k = \frac{z^k}{z^i}$. They are well defined on U_i and satisfy

$$\xi_i^k = \frac{z^k}{z^i} = \frac{z^k}{z^j}\Big/\frac{z^i}{z^j} = \frac{\xi_j^k}{\xi_j^i} \tag{3.5}$$

which is holomorphic on $U_i \cap U_j$ where $\xi_j^i \neq 0$. $\mathbb{P}^n$ is thus a complex manifold. The coordinates $\xi_i = (\xi_i^1, \ldots, \xi_i^n)$ are called *inhomogeneous coordinates*. Alternatively to (3.4) we can also define $\mathbb{P}^n$ as $\mathbb{P}^n = S^{2n+1}/U(1)$, where $U(1)$ acts as $z \to e^{i\phi}z$. This shows that $\mathbb{P}^n$ is compact.

Exercise 3.1: Show that $\mathbb{P}^1 \simeq S^2$ by examining transition functions between the two coordinate patches that one obtains after stereographically projecting the sphere onto $\mathbb{C} \cup \{\infty\}$.

A *complex submanifold* X of a complex manifold M^n is a set $X \subset M^n$ which is given locally as the zeroes of a collection $f_1, \ldots, f_k$ of holomorphic functions such that $\text{rank}(J) \equiv \text{rank}\left(\frac{\partial(f_1,\ldots,f_k)}{\partial(z^1,\ldots,z^n)}\right) = k$. X is a complex manifold of dimension $n-k$, or, equivalently, X has codimension k in M^n. The easiest way to show that X is indeed a complex manifold is to choose local coordinates on M such that X is given by $z^1 = z^2 = \ldots = z^k = 0$. It is then clear that if M is a complex manifold so is X. More generally, if we drop the condition on the rank, we get the definition of an *analytic subvariety*. A point $p \in X$ is a *smooth point* if $\text{rank}(J(p)) = k$. Otherwise p is called a *singular point*. For instance, for $k = 1$, at a smooth point there is no simultaneous solution of $p = 0$ and $dp = 0$.

The importance of projective space, or more generally, of weighted projective space which we will encounter later, lies in the following result: there are no compact complex submanifolds of $\mathbb{C}^n$. This is an immediate consequence of the fact that any global holomorphic function on a compact complex manifold is constant, applied to the coordinate functions (for details, see [25], p.10). This is strikingly different from the real analytic case: any real analytic compact or non-compact manifold can be embedded, by a real analytic embedding, into $\mathbb{R}^N$ for sufficiently large N (Grauert-Morrey theorem).

An *algebraic variety* $X \subset \mathbb{P}^n$ is the zero locus in $\mathbb{P}^n$ of a collection of homogeneous polynomials $\{p_\alpha(z^0, \ldots, z^n)\}$. (A function $f(z)$ is homogeneous of degree d if it satisfies $f(\lambda z) = \lambda^d f(z)$. Taking the derivative w.r.t. λ and setting $\lambda = 1$ at the end, leads to the Euler relation $\sum z^i \partial_i f(z) = d \cdot f(z)$.)

More generally one would consider *analytic varieties*, which are defined in terms of holomorphic functions rather than polynomials. However by the *theorem of Chow* every analytic subvariety of $\mathbb{P}^n$ is in fact algebraic. In more sophisticated mathematical language this means that every analytic subvariety of $\mathbb{P}^n$ is the zero section of some positive power of the universal line bundle over $\mathbb{P}^n$, cf. e.g. [23].

An example of an algebraic submanifold of $\mathbb{P}^4$ is the *quintic hypersurface* which is defined as the zero of the polynomial $p(z) = \sum_{i=0}^4 (z^i)^5$ in $\mathbb{P}^4$. We will see later that this is a three-dimensional Calabi-Yau manifold, and in fact (essentially) the only one that can be written as a hypersurface in $\mathbb{P}^4$, i.e. $X = \{[z^0 : \ldots : z^4] \in \mathbb{P}^4 | p(z) = 0\}$. We can get others by looking at hypersurfaces in products of projective spaces or as complete intersections of more than one hypersurface in higher-dimensional projective spaces and/or products of several projective spaces (here we need several polynomials, homogeneous w.r.t. each $\mathbb{P}^n$). The more interesting generalization is however to enlarge the class of ambient spaces and look at weighted projective spaces.

A weighted projective space is defined much in the same way as a projective space, but with the generalized $\mathbb{C}^*$ action on the homogeneous coordinates

$$\lambda \cdot z = \lambda \cdot (z^0, \ldots, z^n) = (\lambda^{w_0} z^0, \ldots, \lambda^{w_n} z^n) \tag{3.6}$$

where, as before, $\lambda \in \mathbb{C}^*$ and the non-zero integer w_i is called the weight of the homogeneous coordinate z^i. We will consider cases where all weights are positive. However, when one is interested in non-compact situations, one also allows for negative weights. We write $\mathbb{P}^n[w_0, \ldots, w_n] \equiv \mathbb{P}^n[\boldsymbol{w}]$.

Different sets of weights may give isomorphic spaces. A simple example is $\mathbb{P}^n[k\boldsymbol{w}] \simeq \mathbb{P}^n[\boldsymbol{w}]$. One may show that one covers all isomorphism classes if one restricts to so-called *well-formed* spaces [29]. Among the $n+1$ weights of a well-formed space no set of n weights has a common factor. E.g. $\mathbb{P}^2[1, 2, 2]$ is not well formed whereas $\mathbb{P}^2[1, 1, 2]$ is.

Weighted projective spaces are singular, which is most easily demonstrated by means of an example. Consider $\mathbb{P}^2[1, 1, 2]$, i.e. (z^0, z^1, z^2) and $(\lambda z^0, \lambda z^2, \lambda^2 z^2)$ denote the same point. For $\lambda = -1$ the point $[0 : 0 : z^2] \equiv [0 : 0 : 1]$ is fixed but λ acts non-trivially on its neighborhood: we have a $\mathbb{Z}_2$ orbifold singularity at this point. This singularity has locally the form $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the coordinates (x^1, x^2) of $\mathbb{C}^2$ as $\mathbb{Z}_2 : (x^1, x^2) \mapsto -(x^1, x^2)$. In general there is a fixed point for every weight greater than one, a fixed curve for every pair of weights with a common factor greater than one and so on.

A hypersurface $X_d[\boldsymbol{w}]$ in weighted projective space is defined as the vanishing locus of a *quasi-homogeneous* polynomial, $p(\lambda \cdot z) = \lambda^d p(z)$, where d is the degree of $p(z)$, i.e.

$$X_d[\boldsymbol{w}] = \Big\{ [z^0 : \ldots : z^b] \in \mathbb{P}^n[\boldsymbol{w}] \,\Big|\, p(z) = 0 \Big\} \tag{3.7}$$

In this case the Euler relation generalizes to $\sum w_i z^i \partial_i p(z) = d \cdot p(z)$.

Exercise 3.2: Of how many points consist the following 'hypersurfaces'? (1): $(z^0)^2 + (z^1)^2 = 0$ in $\mathbb{P}^1$; (2): $(z^0)^3 + (z^1)^2 = 0$ in $\mathbb{P}^1[2, 3]$. The number of points is equal to the Euler number (the Euler number of a smooth point is one, as can be seen from the Euler formula $\chi = \#\text{vertices} - \#\text{edges} + \#\text{two dimensional faces} \mp \ldots$ of a triangulated space. This also follows from the familiar fact that after removing two points from a sphere with Euler number two one obtains a cylinder whose Euler number is zero).

It can happen that the hypersurface does not pass through the singularities of the ambient space. Take again the example $\mathbb{P}^2[1, 1, 2]$ and consider the quartic hypersurface. At the fixed point $[0 : 0 : z^2]$ only the monomial $(z^2)^2$ survives and the hypersurface constraint would require that $z^2 = 0$. But the point $z^0 = z^1 = z^2 = 0$ is not in $\mathbb{P}^2[1, 1, 2]$. As a second example consider $\mathbb{P}^3[1, 1, 2, 2]$. We now find a singular curve rather than a singular point, namely $z^0 = z^1 = 0$ and a generic hypersurface will intersect this curve in isolated points. To obtain a smooth manifold one has to resolve the singularity, which in this example is a $\mathbb{Z}_2$ singularity. We will not discuss the process of resolution of the singularities but it is mathematically well defined and under control and most efficiently described within the language of toric geometry [30, 31].

Weighted projective spaces are still not the most general ambient spaces one considers in actual string compactifications, in particular when one considers mirror symmetry (see below). The more general concept is that of a toric variety. Toric varieties have some very simple features which allow one to reduce many calculations to combinatorics. Weighted projective spaces are a small subclass of toric varieties. For details we refer to Chapter 7 of [18] and to [31, 32].

We have seen that any complex manifold M can be viewed as a real (analytic) manifold. The tangent space at a point p is denoted by $T_p(M)$ and the tangent bundle by $T(M)$. The *complexified tangent bundle* $T_{\mathbb{C}}(M) = T(M) \otimes \mathbb{C}$ consists of all tangent vectors of M with complex coefficients, i.e. $v = \sum_{j=1}^{2n} v^j \frac{\partial}{\partial x^j}$ with $v^i \in \mathbb{C}$. With the help of (3.2) we can write this as

$$\begin{aligned} v &= \sum_{j=1}^{2n} v^j \frac{\partial}{\partial x^j} = \sum_{j=1}^{n} (v^j + i v^{n+j}) \partial_j + \sum_{j=1}^{n} (v^j - i v^{n+j}) \bar{\partial}_j \\ &\equiv v^{1,0} + v^{0,1} \end{aligned} \tag{3.8}$$

We have thus a decomposition

$$T_{\mathbb{C}}(M) = T^{1,0}(M) \oplus T^{0,1}(M) \tag{3.9}$$

into *vectors of type* $(1,0)$ and of type $(0,1)$: $T^{1,0}(M)$ is spanned by $\{\partial_i\}$ and $T^{0,1}(M)$ by $\{\bar{\partial}_i\}$. Note that $T_p^{0,1}(M) = \overline{T_p^{1,0}(M)}$ and that the splitting into the two subspaces is preserved under holomorphic coordinate changes. The transition functions of $T^{1,0}(M)$ are holomorphic, and we therefore call it the holomorphic tangent bundle. A holomorphic section of $T^{1,0}(M)$ is called a *holomorphic vector field*; its component functions are holomorphic.

$T^{1,0}(M)$ is just one particular example of a *holomorphic vector bundle* $E \xrightarrow{\pi} M$. Holomorphic vector bundles of rank k are characterized by their holomorphic transition functions which are elements of $Gl(k, \mathbb{C})$ (rather than $Gl(n, \mathbb{R})$ as in the real case) with holomorphic matrix elements.

In the same way as in (3.9) we decompose the dual space, the space of one-forms:

$$T^*_{\mathbb{C}}(M) = T^{*1,0}(M) \oplus T^{*0,1}(M)\,. \tag{3.10}$$

$T^{*1,0}(M)$ and $T^{*0,1}(M)$ are spanned by $\{dz^i\}$ and $\{d\bar{z}^i\}$, respectively. By taking tensor products we can define differential forms of type (p,q) as sections of $\overset{p}{\wedge} T^{*1,0}(M) \overset{q}{\wedge} T^{*0,1}(M)$. The space of (p,q)-forms will be denoted by $A^{p,q}$. Clearly $\overline{A^{p,q}} = A^{q,p}$. If we denote the space of sections of $\overset{r}{\wedge} T^*_{\mathbb{C}}(M)$ by A^r, we have the decomposition

$$A^r = \bigoplus_{p+q=r} A^{p,q}\,. \tag{3.11}$$

This decomposition is independent of the choice of local coordinate system.

Using the underlying real analytic structure we can define the exterior derivative d. If $\omega \in A^{p,q}$, then

$$d\omega \in A^{p+1,q} \oplus A^{p,q+1}\,. \tag{3.12}$$

We write $d\omega = \partial\omega + \bar{\partial}\omega$ with $\partial\omega \in A^{p+1,q}$ and $\bar{\partial}\omega \in A^{p,q+1}$. This defines the two operators

$$\partial :\, A^{p,q} \to A^{p+1,q}\,, \qquad \bar{\partial} :\, A^{p,q} \to A^{p,q+1}\,, \tag{3.13}$$

and

$$d = \partial + \bar{\partial}\,. \tag{3.14}$$

The following results are easy to verify:

$$d^2 = (\partial + \bar{\partial})^2 \equiv 0 \quad \Rightarrow \quad \partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0\,. \tag{3.15}$$

Here we used that $\partial^2 : A^{p,q} \to A^{p+2,q}$, $\bar\partial^2 : A^{p,q} \to A^{p,q+2}$, $(\partial\bar\partial + \bar\partial\partial) : A^{p,q} \to A^{p+1,q+1}$, i.e. that the three operators map to three different spaces. They must thus vanish separately.

Eq. (3.14) is not true on an almost complex manifold. Alternative to the way we have defined complex structures we could have started with an almost complex structure – a differentiable isomorphism $J : T(M) \to T(M)$ with $J^2 = -\mathbb{1}$ – such that the splitting (3.9) of $T(M)$ is into eigenspaces of J with eigenvalues $+i$ and $-i$, respectively. Then (3.12) would be replaced by $d\omega \in A^{p+2,q-1} \oplus A^{p+1,q} \oplus A^{p,q+1} \oplus A^{p-1,q+2}$ and (3.14) by $d = \partial + \bar\partial + \ldots$. Only if the almost complex structure satisfies an integrability condition – the vanishing of the Nijenhuis tensor – do (3.12) and (3.14) hold. A theorem of Newlander and Nierenberg then guarantees that we can construct on M an atlas of holomorphic charts and M is a complex manifold in the sense of the definition that we have given, see e.g. [13, 25].

ω is called a *holomorphic p-form* if it is of type $(p,0)$ and $\bar\partial\omega = 0$, i.e. if it has holomorphic coefficient functions. Likewise $\bar\omega$ of type $(0,q)$ with $\bar\partial\bar\omega = 0$ is called *anti-holomorphic*. $\Omega^p(M)$ denotes the vector-space of holomorphic p-forms. We leave it as an exercise to write down the explicit expressions, in terms of coefficients, of $\partial\omega$, etc.

3.2 Kähler manifolds

The next step is to introduce additional structures on a complex manifold: a hermitian metric and a hermitian connection.

A *hermitian metric* is a covariant tensor field of the form $\sum_{i,j=1}^n g_{i\bar\jmath} dz^i \otimes d\bar z^j$, where $g_{i\bar\jmath} = g_{i\bar\jmath}(z)$ (here the notation is not to indicate that the components are holomorphic functions; they are not!) such that $g_{j\bar\imath}(z) = \overline{g_{i\bar\jmath}(z)}$ and $g_{i\bar\jmath}(z)$ is a positive definite matrix, that is, for any $\{v^i\} \in \mathbb{C}^n$, $v^i g_{i\bar\jmath} \bar v^{\bar\jmath} \geq 0$ with equality only if all $v^i = 0$. To any hermitian metric we associate a $(1,1)$-form

$$\omega = i \sum_{i,j=1}^n g_{i\bar\jmath} dz^i \wedge d\bar z^j \,. \tag{3.16}$$

ω is called the *fundamental form* associated with the hermitian metric g.

Exercise 3.3: Show that ω is a real $(1,1)$-form, i.e. that $\omega = \bar\omega$.

One can introduce a hermitian metric on any complex manifold (see e.g. [26], p. 145).

Exercise 3.4: Show that

$$\begin{aligned}
\frac{\omega^n}{n!} &= (i)^n g(z) dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^n \wedge d\bar z^n \\
&= (i)^n (-1)^{n(n-1)/2} g(z) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar z^1 \wedge \cdots \wedge d\bar z^n \\
&= 2^n g(z) dx^1 \wedge \cdots \wedge dx^{2n}
\end{aligned} \tag{3.17}$$

where $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}}$ and $g = \det(g_{i\bar\jmath}) > 0$. ω^n is thus a good volume form on M. This shows once more that complex manifolds always possess an orientation.

The inverse of the hermitian metric is $g^{i\bar\jmath}$ which satisfies $g^{j\bar\imath} g_{j\bar k} = \delta^{\bar\imath}_{\bar k}$ and $g_{i\bar\jmath} g^{k\bar\jmath} = \delta_i^k$ (summation convention used). We use the metric to raise and lower indices, whereby they change their type. Note that under holomorphic coordinate changes, the index structure of the metric is preserved, as is that of any other tensor field.

A hermitian metric g whose associated fundamental form ω is closed, i.e. $d\omega = 0$, is called a *Kähler metric*. A complex manifold endowed with a Kähler metric is called a *Kähler manifold*. ω is the *Kähler form* . An immediate consequence of $d\omega = 0 \Rightarrow \partial\omega = \bar\partial\omega = 0$ is

$$\partial_i g_{j\bar k} = \partial_j g_{i\bar k} \,, \qquad \bar\partial_i g_{j\bar k} = \bar\partial_k g_{j\bar\imath} \qquad \text{(Kähler condition)} \,. \tag{3.18}$$

From this one finds immediately that the only non-zero coefficients of the Riemannian connection are

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}\,, \qquad \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}} = g^{l\bar{k}}\bar{\partial}_{\bar{\imath}} g_{l\bar{\jmath}}\ . \tag{3.19}$$

The vanishing of the connection coefficients with mixed indices is a necessary and sufficient condition that under parallel transport the holomorphic and the anti-holomorphic tangent spaces do not mix (see below).

Note that while all complex manifolds admit a hermitian metric, this does not hold for Kähler metrics. Counterexamples are quaternionic manifolds which appear as moduli spaces of type II compactifications on Calabi-Yau manifolds. Another example is $S^{2p+1} \otimes S^{2q+1}$, $q > 1$. A complex submanifold X of a Kähler manifold M is again a Kähler manifold, with the induced Kähler metric. This follows easily if one goes to local coordinates on M where X is given by $z^1 = \ldots = z^k = 0$.

From (3.18) we also infer the *local* existence of a real *Kähler potential* K in terms of which the Kähler metric can be written as

$$g_{i\bar{\jmath}} = \partial_i \bar{\partial}_j K \tag{3.20}$$

or, equivalently, $\omega = i\partial\bar{\partial}K$. The Kähler potential is not uniquely defined: $K(z,\bar{z})$ and $K(z,\bar{z})+f(z)+\bar{f}(\bar{z})$ lead to the same metric if f and $\bar{f}$ are holomorphic and anti-holomorphic functions (on the patch on which K is defined), respectively.

From now on, unless stated otherwise, we will restrict ourselves to Kähler manifolds; some of the results are, however, true for arbitrary complex manifolds. Also, if in doubt, assume that the manifold is compact.

Exercise 3.5: Determine a *hermitian connection* by the two requirements: (1) The only non-vanishing coefficients are Γ^i_{jk} and $\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}$ and (2) $\nabla_{\bar{\imath}} g_{j\bar{k}} = 0$. Show that the connection is torsionfree, i.e. $T^k_{ij} \equiv \Gamma^k_{ij} - \Gamma^k_{ji} = 0$ if g is a Kähler metric. Check that the connection so obtained is precisely the Riemannian connection, i.e. the hermitian and the Riemannian structures on a Kähler manifold are compatible.

Exercise 3.6: Derive the components of the Riemann tensor on a Kähler manifold. Show that the only non-vanishing components of the Riemann tensor are those with the index structure $R_{i\bar{\jmath}k\bar{l}}$ and those related by symmetries. In particular the components of the type R_{ij**} are zero. Show that the non-vanishing components are

$$R_{i\bar{\jmath}k\bar{l}} = -\partial_i\bar{\partial}_{\bar{\jmath}} g_{k\bar{l}} + g^{m\bar{n}}(\partial_i g_{k\bar{n}})(\bar{\partial}_{\bar{\jmath}} g_{m\bar{l}})\ . \tag{3.21}$$

Here the sign conventions are such that $[\nabla_i, \nabla_{\bar{\jmath}}]V_k = -R_{i\bar{\jmath}k}{}^l V_l$.

Exercise 3.7: The Ricci tensor is defined as $R_{i\bar{\jmath}} \equiv g^{k\bar{l}} R_{i\bar{\jmath}k\bar{l}}$. Prove that

$$R_{i\bar{\jmath}} = -\partial_i \bar{\partial}_{\bar{\jmath}}(\log\det g)\ . \tag{3.22}$$

Show that this is the same (up to a sign) as $R_{i\mu\bar{\jmath}}{}^{\mu} = R_{i\mu\bar{\jmath}\nu} g^{\mu\nu}$, $\mu = (k, \bar{k})$.

One also defines the *Ricci-form* (of type $(1,1)$) as

$$\mathcal{R} = iR_{j\bar{k}}dz^j \wedge d\bar{z}^k = -i\partial\bar{\partial}\log(\det\, g) \tag{3.23}$$

which satisfies $d\mathcal{R} = 0$. Note that $\log(\det\, g)$ is not a globally defined function since $\det g$ transforms as a density under change of coordinates. $\mathcal{R}$ is however globally defined (why?).

We learn from (3.23) that the Ricci form depends only on the volume form of the Kähler metric and on the complex structure (through ∂ and $\bar{\partial}$). Under a change of metric, $g \to g'$, the Ricci form changes as

$$\mathcal{R}(g') = \mathcal{R}(g) - i\partial\bar{\partial}\log\left(\frac{\det(g'_{k\bar{l}})}{\det(g_{k\bar{l}})}\right)\ , \tag{3.24}$$

where the ratio of the two determinants is a globally defined non-vanishing function on M.

Example 3.3: Complex projective space. To demonstrate that it is a Kähler manifold we give an explicit metric, the so called *Fubini-Study metric*. Recall that $\mathbb{P}^n = \{[z^0 : \ldots : z^n]; 0 \neq (z^0 : \ldots : z^n) \in \mathbb{C}^{n+1}\}$ and $U_0 = \{[1, z^1 : \ldots : z^n]\} \simeq \mathbb{C}^n$ is an open subset of $\mathbb{P}^n$. Set

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \log(1 + |z^1|^2 + \ldots + |z^n|^2) \equiv \partial_i \partial_{\bar{\jmath}} \ln(1 + |z|^2) \tag{3.25}$$

or, equivalently,

$$\omega = i\partial\bar{\partial} \log(1 + |z|^2) = i \left(\frac{dz^i \wedge d\bar{z}^i}{1 + |z|^2} - \frac{\bar{z}^i dz^i \wedge z^j d\bar{z}^j}{(1 + |z|^2)^2} \right) \tag{3.26}$$

Closure of ω is obvious if one uses (3.15). From (3.25) we also immediately read off the Kähler potential of the Fubini-Study metric (cf. (3.20)) on U_0. Clearly this is only defined locally.

Exercise 3.8: Show that for any non-zero vector u, $u^i g_{i\bar{\jmath}} \bar{u}^{\bar{\jmath}} \geq 0$ to prove positive definiteness of the Fubini-Study metric.

On the other hand, ω is globally defined on $\mathbb{P}^n$. To see this, let $U_1 = \{(w^0, 1, w^2, \ldots, w^n)\} \subset \mathbb{P}^n$ and check what happens to ω on the overlap $U_0 \cap U_1 = \{[1 : z^1 : \cdots : z^n] = [w^0 : 1 : w^2 : \cdots : w^n]\}$, where $z^i = \frac{w^i}{w^0}$, for all $i \neq 1$ and $z^1 = \frac{1}{w^0}$. Then

$$\begin{aligned} \omega &= i\partial\bar{\partial} \log(1 + |z^1|^2 + \cdots + |z^n|^2) = i\partial\bar{\partial} \log \left(1 + \frac{1}{|w^0|^2} + \sum_{i=2}^{n} \frac{|w^i|^2}{|w^0|^2} \right) \\ &= i \Big(\partial\bar{\partial} \log(1 + |w|^2) - \partial\bar{\partial} \log(|w^0|^2) \Big) = i\partial\bar{\partial} \log(1 + |w|^2) \end{aligned} \tag{3.27}$$

since w^0 is holomorphic on $U_0 \cap U_1$. So ω and the corresponding Kähler metric are globally defined. Complex projective space is thus a Kähler manifold and so is every complex submanifold. With[3]

$$\det(g_{i\bar{\jmath}}) = \frac{1}{(1 + |z|^2)^{n+1}} \tag{3.28}$$

one finds

$$R_{i\bar{\jmath}} = -\partial_i \bar{\partial}_{\bar{\jmath}} \log \left(\frac{1}{(1 + |z|^2)^{n+1}} \right) = (n + 1) g_{i\bar{\jmath}} \tag{3.29}$$

which shows that the Fubini-Study metric is a *Kähler-Einstein metric* and $\mathbb{P}^n$ a *Kähler-Einstein manifold.*

3.3 Holonomy group of Kähler manifolds

The next topic we want to discuss is the *holonomy group of Kähler manifolds.* Recall that the holonomy group of a Riemannian manifold of (real) dimension m is a subgroup of $O(m)$. It follows immediately from the index structure of the connection coefficients of a Kähler manifold that under parallel transport elements of $T^{1,0}(M)$ and $T^{0,1}(M)$ do not mix. Since the length of a vector does not change under parallel transport, the holonomy group of a Kähler manifold is a subgroup of $U(n)$ where n is the complex dimension of the manifold. [4] In particular, elements of $T^{1,0}(M)$ transform as $\boldsymbol{n}$ and elements of $T^{0,1}(M)$ as $\overline{\boldsymbol{n}}$ of $U(n)$. Consider now parallel transport around an infinitesimal loop in the (μ, ν)-plane with area $\delta a^{\mu\nu} = -\delta a^{\nu\mu}$. Under parallel transport around this loop a vector V changes by an amount δV given in (A.14). In complex coordinates this is $\delta V^i = -\delta a^{k\bar{l}} R_{k\bar{l}}{}^i{}_j V^j$. From what we said above it follows that on a Kähler manifold the matrix $-\delta a^{k\bar{l}} R_{k\bar{l}i}{}^j$ must be an element of the Lie algebra $u(n)$. The trace of this matrix, which is proportional to the Ricci tensor, generates the $u(1)$ part in the decomposition

[3] To show this, use $\det(\delta_{ij} - v_i v_{\bar{\jmath}}) = \exp\left(\operatorname{tr} \log(\delta_{ij} - v_i v_{\bar{\jmath}})\right) = \exp\left(\log(1 - |v|^2)\right) = (1 - |v|^2)$.

[4] The unitary group $U(n)$ is the set of all complex $n \times n$ matrices which leave invariant a hermitian metric $\overline{g_{i\bar{\jmath}}} = g_{j\bar{\imath}}$, i.e. $UgU^\dagger = g$. For the choice $g_{i\bar{\jmath}} = \delta_{ij}$ one obtains the familiar condition $UU^\dagger = \mathbb{1}$.

$u(n) \simeq su(n) \oplus u(1)$. We thus learn that the holonomy group of a Ricci-flat Kähler manifold is a subgroup of $SU(n)$. Conversely, one can show that any $2n$-dimensional manifold with $U(n)$ holonomy admits a Kähler metric and if it has $SU(n)$ holonomy it admits a Ricci-flat Kähler metric. This uses the fact that holomorphic and anti-holomorphic indices do not mix, which implies that all connection coefficients with mixed indices must vanish. One then proceeds with the explicitly construction of an almost complex structure with vanishing Nijenhuis tensor. Details can be found in [4, 28].

We should mention that strictly speaking the last argument is only valid for the restricted holonomy group $\mathcal{H}_0$ (which is generated by parallel transport around contractible loops). Also, in general only the holonomy around *infinitesimal* loops is generated by the Riemann tensor. For finite (but still contractible) loops, derivatives of the Riemann tensor of arbitrary order will appear [12]. For Kähler manifolds we do however have the $U(n)$ invariant split of the indices $\mu = (i, \bar{\imath})$ and $U(n)$ is a maximal compact subgroup of $SO(2n)$. Thus the restricted holonomy group is not bigger than $U(n)$ For simply connected manifolds the restricted holonomy group is already the full holonomy group. For non-simply connected manifolds the full holonomy group and the restricted holonomy group may differ. Their quotient is countable and the restricted holonomy group is the identity component of the full holonomy group, i.e. for a generic Riemannian manifold it is $SO(m)$ (cf. [12]).

3.4 Cohomology of Kähler manifolds

Before turning to the next subject, *homology and cohomology* on complex manifolds, we will give a very brief and incomplete summary of these concepts in the real situation, which, of course, also applies to complex manifolds, if they are viewed as real analytic manifolds.

On a smooth, connected manifold M one defines p-chains a_p as formal sums $a_p = \sum_i c_i N_i$ of p-dimensional oriented submanifolds on M. If the coefficients c_i are real (complex, integer), one speaks of real (complex, integral) chains. Define ∂ as the operation of taking the boundary with the induced orientation. $\partial a \equiv \sum c_i \partial N_i$ is then a $p-1$-chain. Let $Z_p = \{a_p | \partial a_p = \emptyset\}$ be the set of *cycles*, i.e. the set of chains without boundary and let $B_p = \{\partial a_{p+1}\}$ be the set of *boundaries*. Since $\partial\partial a_p = \emptyset$, $B_p \subset Z_p$. The *p-th homology group of M* is defined as

$$H_p = Z_p / B_p \,. \tag{3.30}$$

Depending on the coefficient group one gets $H_p(M, \mathbb{R})$, $H_p(M, \mathbb{C})$, $H_p(M, \mathbb{Z})$, etc. Elements of H_p are equivalence classes of cycles $z_p \simeq z_p + \partial a_{p+1}$, called *homology classes* and denoted by $[z_p]$.

One version of *Poincaré duality* is the following isomorphism between homology groups, valid on orientable connected smooth manifolds of real dimension m:

$$H_r(M, \mathbb{R}) \simeq H_{m-r}(M, \mathbb{R}) \,. \tag{3.31}$$

One defines the *r-th Betti number* b_r as

$$b_r = \dim(H_r(M, \mathbb{R})) \,. \tag{3.32}$$

As a consequence of Poincaré duality,

$$b_r(M) = b_{m-r}(M) \,. \tag{3.33}$$

We now turn to de Rham cohomology, which is defined with the exterior derivative operator $d : A^r \to A^{r+1}$. Let Z^p be the set of *closed p-forms*, i.e. $Z^p = \{\omega_p | d\omega_p = 0\}$ and let B^p be the set of *exact p-forms* $B^p = \{d\omega_{p-1}\}$. The *de Rham cohomology groups* H^p are defined as the quotients

$$H^p_{\text{D.R.}} = Z^p / B^p \,. \tag{3.34}$$

Elements of H^p are equivalence classes of closed forms $\omega_p \simeq \omega_p + d\alpha_{p-1}$, called cohomology classes and denoted by $[\omega_p]$. Each equivalence class possesses one harmonic representative, i.e. a zero mode of the Laplacian $\Delta = dd^* + d^*d$. The action of Δ on p-forms is

$$\Delta\omega_{\mu_1\cdots\mu_p} = -\nabla^\nu\nabla_\nu\omega_{\mu_1\cdots\mu_p} - pR_{\nu[\mu_1}\omega^\nu{}_{\mu_2\cdots\mu_p]} - \frac{1}{2}p(p-1)R_{\nu\rho[\mu_1\mu_2}\omega^{\nu\rho}{}_{\mu_3\cdots\mu_p]}\,. \tag{3.35}$$

Since the number of (normalizable) harmonic forms on a compact manifold is finite, the Betti numbers are all finite.

Exercise 3.9: Derive (3.35).

Given both the homology and the cohomology classes, we can define an inner product

$$\pi(z_p, \omega_p) = \int_{z_p} \omega_p\,, \tag{3.36}$$

where $\pi(z_p, \omega_p)$ is called a *period* (of ω_p). We speak of an *integral cohomology class* $[\omega_p] \in H_{\text{D.R.}}(M, \mathbb{Z})$ if the period over any integral cycle is integer.

Exercise 3.10: Prove, using Stoke's theorem, that the integral does not depend on which representatives of the two classes are chosen.

A theorem of de Rham ensures that the above inner product between homology and cohomology classes is bilinear and non-degenerate, thus establishing an isomorphism between homology and cohomology. The following two facts are consequences of de Rham's theorem:

(1) Given a basis $\{z_i\}$ for H_p and any set of periods ν_i, $i = 1, \ldots, b_p$, there exists a closed p-from ω such that $\pi(z_i, \omega) = \nu_i$.

(2) If all periods of a p-form vanish, ω is exact.

Another consequence of de Rham's theorem is the following important result: Given any p-cycle z there exists a closed $(m-p)$-form α, called the *Poincaré dual of* z such that for any closed p-form ω

$$\int_z \omega = \int_M \alpha \wedge \omega\,. \tag{3.37}$$

Since ω is closed, α is only defined up to an exact form. In terms of their Poincaré duals α and β we can define the *intersection number* $A \cdot B$ between a p-cycle A and an $(m-p)$-cycle B as

$$A \cdot B = \int_M \alpha \wedge \beta\,. \tag{3.38}$$

This notion is familiar from Riemann surfaces.

So much for the collection of some facts about homology and cohomology on real manifolds. They are also valid on complex manifolds if one views them as real analytic manifolds. However one can use the complex structure to define (among several others) the so-called *Dolbeault cohomology* or $\bar\partial$-cohomology. As the (second) name already indicates, it is defined w.r.t. the operator $\bar\partial : A^{p,q}(M) \to A^{p,q+1}(M)$. A (p,q)-form α is *$\bar\partial$-closed* if $\bar\partial\alpha = 0$. The space of $\bar\partial$-closed (p,q)-forms is denoted by $Z^{p,q}_{\bar\partial}$. A (p,q)-form β is *$\bar\partial$-exact* if it is of the form $\beta = \bar\partial\gamma$ for $\gamma \in A^{p,q-1}$. Since $\bar\partial^2 = 0$, $\bar\partial(A^{p,q}(M)) \subset Z^{p,q+1}_{\bar\partial}(M)$. The Dolbeault cohomology groups are then defined as

$$H^{p,q}_{\bar\partial}(M) = \frac{Z^{p,q}_{\bar\partial}(M)}{\bar\partial(A^{p,q-1}(M))}\,. \tag{3.39}$$

There is a lemma (by Dolbeault) analogous to the Poincaré-lemma, which ensures that the Dolbeault cohomology groups (for $q \geq 1$) are locally[5] trivial. This is also referred to as the *$\bar\partial$-Poincaré lemma.*

[5]More precisely, on polydiscs $P_r = \{z \in \mathbb{C}^n | z^i| < r,\ \text{for all } i = 1, \ldots, n\}$.

The dimensions of the (p,q) cohomology groups are called *Hodge numbers*

$$h^{p,q}(M) = \dim_{\mathbb{C}}(H^{p,q}_{\bar\partial}(M))\,. \tag{3.40}$$

They are finite for compact, complex manifolds [23]. The Hodge numbers of a Kähler manifold are often arranged in the *Hodge diamond*:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{3.41}$$

which we have displayed here for a three complex dimensional Kähler manifold. We will later show that for a Calabi-Yau manifold of the same dimension the only independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$.

We can now define a scalar product between two forms, φ and ψ, of type (p,q):[6]

$$\psi = \frac{1}{p!q!}\psi_{i_1\dots i_p\bar\jmath_1\dots\bar\jmath_q}(z)dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q} \tag{3.42}$$

and likewise for φ. First define

$$(\varphi,\psi)(z) = \frac{1}{p!q!}\varphi_{i_1\dots i_p\bar\jmath_1\dots\bar\jmath_q}(z)\bar\psi^{i_1\cdots{}_p\bar\jmath_1\cdots\bar\jmath_q}(z) \tag{3.43}$$

where

$$\bar\psi^{i_1\dots i_p\bar\jmath_1\dots\bar\jmath_q}(z) = g^{i_1\bar k_1}\cdots g^{i_p\bar k_p}g^{l_1\bar\jmath_1}\cdots g^{l_q\bar\jmath_q}\overline{\psi_{k_1\dots k_p\bar l_1\dots\bar l_q}(z)}\,. \tag{3.44}$$

Later we will also need the definition

$$\bar\psi = \frac{1}{p!q!}\overline{\psi_{i_1\dots i_p\bar\jmath_1\dots\bar\jmath_q}dz^{i_1}\wedge\cdots\wedge d\bar z^{j_q}} = \frac{1}{p!q!}\bar\psi_{j_i\dots j_q\bar\imath_1\dots\bar\imath_p}dz^{j_1}\wedge\cdots\wedge d\bar z^{i_p}\,, \tag{3.45}$$

where

$$\overline{\psi_{k_1\dots k_p\bar l_1\dots\bar l_q}} = (-1)^{pq}\bar\psi_{l_1\dots l_q\bar k_1\dots\bar k_p}\,. \tag{3.46}$$

The inner product $(\ ,\ ):\ A^{p,q}\times A^{p,q}\to\mathbb{C}$ is then

$$(\varphi,\psi) = \int_M(\varphi,\psi)(z)\frac{\omega^n}{n!}\,. \tag{3.47}$$

The following two properties are easy to verify:

$$\begin{aligned}&(\psi,\varphi) = \overline{(\varphi,\psi)}\ ,\\ &(\varphi,\varphi)\geq 0 \quad \text{with equality only for } \varphi=0\ .\end{aligned} \tag{3.48}$$

We define the Hodge-$*$ operator $*: A^{p,q}\to A^{n-q,n-p}$, $\psi\mapsto *\psi$ by requiring[7]

$$(\varphi,\psi)(z)\frac{\omega^n}{n!} = \varphi(z)\wedge *\bar\psi(z)\,. \tag{3.49}$$

[6] A good and detailed reference for the following discussion is the third chapter of [26].

[7] Note that there are several differing notations in the literature; e.g. Griffiths and Harris define an operator $*_{\rm GH}\ :\ A^{p,q}\to A^{n-p,n-q}$. What they call $*_{\rm GH}\ \psi$ we have called $*\bar\psi$.

Exercise 3.11: Show that for $\psi \in A^{p,q}$,

$$
\begin{aligned}
*\psi &= \tfrac{(i)^n(-1)^{n(n-1)/2+np}}{p!q!(n-p)!(n-q)!} g\epsilon^{m_1\dots m_p}{}_{\bar{j}_1\dots\bar{j}_{n-p}}\epsilon^{\bar{n}_1\dots\bar{n}_q}{}_{l_1\dots l_{n-q}} \qquad (3.50)\\
&\quad \cdot \psi_{m_1\dots m_p\bar{n}_1\dots\bar{n}_q} dz^{l_1}\wedge\cdots\wedge dz^{l_{n-q}}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_{n-p}} \in A^{n-q,n-p} \ .
\end{aligned}
$$

Here we defined $\epsilon_{i_1\dots i_n} = \pm 1$ and its indices are raised with the metric, as usual; i.e. $\epsilon^{\bar{j}_1\cdots\bar{j}_n} = \pm g^{-1}$.

Exercise 3.12: Prove the following properties of the $*$-operator:

$$
\begin{aligned}
\psi &= \overline{\psi} \ ,\\
**\psi &= (-1)^{p+q}\,\psi \quad , \quad \psi \in A^{p,q} \ . \qquad (3.51)
\end{aligned}
$$

Exercise 3.13: For ω the fundamental form and α an arbitrary *real* $(1,1)$-form, derive the following two identities, valid on a three-dimensional Kähler manifold:

$$
\begin{aligned}
*\omega &= \frac{1}{2}\omega\wedge\omega \ ,\\
\alpha &= \frac{3}{2}\frac{\int \alpha\wedge\omega\wedge\omega}{\int\omega\wedge\omega\wedge\omega}\cdot\omega\wedge\omega - \alpha\wedge\omega \ . \qquad (3.52)
\end{aligned}
$$

Exercise 3.14: Show that on a three-dimensional complex manifold for $\Omega \in A^{3,0}$ and $\alpha \in A^{2,1}$,

$$
\begin{aligned}
*\Omega &= i\Omega \,,\\
*\,\alpha &= -i\alpha \,. \qquad (3.53)
\end{aligned}
$$

Given the scalar product (3.47), we can define the adjoint of the $\bar\partial$ operator, $\bar\partial^* : A^{p,q}(M) \to A^{p,q-1}(M)$ via

$$
(\bar\partial^*\psi,\varphi) = (\psi,\bar\partial\varphi)\,, \qquad \forall\varphi\in A^{p,q-1}(M)\,. \qquad (3.54)
$$

Exercise 3.15: Show that on M compact,

$$
\bar\partial^* = -*\partial* \ . \qquad (3.55)
$$

Exercise 3.16: Show that, given a (p,q)-form ψ,

$$
(\bar\partial^*\psi)_{i_1\dots i_p\bar{j}_2\dots\bar{j}_q} = (-1)^{p+1}\nabla^{\bar{j}_1}\psi_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} \ . \qquad (3.56)
$$

We now define the *$\bar\partial$-Laplacian* as

$$
\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial\,, \qquad \Delta_{\bar\partial} : A^{p,q}(M) \to A^{p,q}(M) \qquad (3.57)
$$

and call ψ a $(\bar\partial-)$*harmonic form* if it satisfies

$$
\Delta_{\bar\partial}\psi = 0\,. \qquad (3.58)
$$

The space of harmonic (p,q)-forms on M is denoted by $\mathcal{H}^{p,q}(M)$.

Exercise 3.17: Show that on a compact manifold, ψ is harmonic iff $\bar\partial\psi = \bar\partial^*\psi = 0$, i.e. a harmonic form has zero curl and zero divergence with respect to its anti-holomorphic indices. Show furthermore that a harmonic form is orthogonal to any exact form and is therefore never exact.

In analogy to de Rham cohomology, one has the (complex version of the) *Hodge Theorem*: $A^{p,q}$ has a unique orthogonal decomposition

$$A^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\partial} A^{p,q-1} \oplus \bar{\partial}^* A^{p,q+1} \,. \tag{3.59}$$

In other words, every $\varphi \in A^{p,q}$ has a unique decomposition

$$\varphi = h + \bar{\partial}\psi + \bar{\partial}^* \eta \tag{3.60}$$

where $h \in \mathcal{H}^{p,q}$, $\psi \in A^{p,q-1}$ and $\eta \in A^{p,q+1}$. If $\bar{\partial}\varphi = 0$ then $\bar{\partial}^*\eta = 0$,[8] i.e. we have the unique decomposition of $\bar{\partial}$-closed forms

$$Z^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q} \oplus \bar{\partial} A^{p,q-1} \tag{3.61}$$

With reference to (3.39) we have thus shown that

$$H^{p,q}_{\bar{\partial}}(M) \simeq \mathcal{H}^{p,q}(M) \tag{3.62}$$

or, in words, every $\bar{\partial}$-cohomology class of (p,q)-forms has a unique harmonic representative $\in \mathcal{H}^{p,q}$. Conversely, every harmonic form defines a cohomology class.

The *Kähler class* of a Kähler form ω is the set of Kähler forms belonging to the cohomology class $[\omega]$ of ω.

Exercise 3.18: Prove that the Kähler form is harmonic.

In addition to the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}}$, one defines two further Laplacians on a complex manifold: $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and the familiar $\Delta_d = dd^* + d^*d$. The importance of the Kähler condition is manifest in the following result which is valid on Kähler manifolds but not generally on complex manifolds:

$$\Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta_d \tag{3.63}$$

i.e. the $\bar{\partial}-$, $\partial-$ and $d-$harmonic forms coincide. An elementary proof of (3.63) proceeds by working out the three Laplacians in terms of covariant derivatives and Riemann tensors on a Kähler manifold. For other proofs, see e.g. [23].

One immediate consequence of (3.63) is that Δ_d does not change the index type of a form. Another important consequence is that on Kähler manifolds every holomorphic p-form is harmonic and vice-versa, every harmonic $(p,0)$ form is holomorphic. Indeed, if $\alpha \in \Omega^p \subset A^{p,0}$, $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$. The latter is true since $\bar{\partial}^* : A^{p,q} \to A^{p,q-1}$ and can also be seen directly from (3.56). Conversely, $\Delta\alpha = 0$ implies $\bar{\partial}\alpha = 0$ which, for $\alpha \in \mathcal{H}^{p,0}$, means $\alpha \in \Omega^p$.

It follows from (3.63) that on Kähler manifolds

$$\begin{aligned} \sum_{p+q=r} h^{p,q} &= b_r \,, \\ \sum_{p,q} (-1)^{p+q} h^{p,q} &= \sum_r (-1)^r b_r = \chi(M) \,, \end{aligned} \tag{3.64}$$

where $\chi(M)$ is the Euler number of M. The decomposition of the Betti numbers into Hodge numbers corresponds to the $U(n)$ invariant decomposition $\mu = (i, \bar{\imath})$. The second relation also holds in the non-Kähler case where the first relation is replaced by an inequality ($\geq$); i.e. the decomposition of forms (3.11) does not generally carry over to cohomology. Note that (3.64) relates real and complex dimensions.

In general, the Hodge numbers do depend on the complex structure. On compact manifolds which admit a Kähler metric, these numbers do however not change under *continuous* deformations of the complex structure. They also do not depend on the metric. What does depend on the metric is the

[8] From $\bar{\partial}\varphi = \bar{\partial}\bar{\partial}^*\eta$ it follows that $(\bar{\partial}\varphi, \eta) = (\bar{\partial}\bar{\partial}^*\eta, \eta) = (\bar{\partial}^*\eta, \bar{\partial}^*\eta)$.

harmonic representative of each class, but the difference between such harmonic representatives is always an exact form.

The Hodge numbers of Kähler manifolds are not all independent. From $\overline{A^{p,q}} = A^{q,p}$ we learn

$$h^{p,q} = h^{q,p} \,. \tag{3.65}$$

This symmetry ensures that all odd Betti numbers of Kähler manifolds are even (possibly zero). Furthermore, since $[\Delta_d, *] = 0$ and since $* : A^{p,q} \to A^{n-q,n-p}$ we conclude

$$h^{p,q} = h^{n-q,n-p} \overset{(3.65)}{=} h^{n-p,n-q} \,. \tag{3.66}$$

The existence of a closed $(1,1)$-form, the Kähler form ω (which is in fact harmonic, cf. Exercise 3.18), ensures that

$$h^{p,p} > 0 \quad \text{for} \quad p = 0, \dots, n \,. \tag{3.67}$$

Indeed, $\omega^p \in H^{p,p}(M)$ is obviously closed. If it were exact for some p, then ω^n were also exact. But this is impossible since ω^n is a volume form. $h^{0,0} = 1$ if the manifold is connected. The elements of $H^{0,0}(M, \mathbb{C})$ are the complex constants. One can show that on $\mathbb{P}^n$ the Kähler form generates the whole cohomology, i.e. $h^{p,p}(\mathbb{P}^n) = 1$ for $p = 0, \dots n$, with all other Hodge numbers vanishing.

For instance, on a connected three-dimensional Kähler manifold, these symmetries leave only five independent Hodge numbers, e.g. $h^{1,0}$, $h^{2,0}$, $h^{1,1}$, $h^{2,1}$ and $h^{3,0}$. For Ricci-flat Kähler manifolds, which we will consider in detail below, we will establish three additional restrictions on its Hodge numbers.

We have already encountered one important cohomology class on Kähler manifolds: from (3.23) we learn that $\mathcal{R} \in H^{1,1}(M, \mathbb{C})$ and from (3.24) that under change of metric $\mathcal{R}$ varies within a given cohomology class. In fact, one can show that, if properly normalized, the Ricci form defines an element on $H^{1,1}(M, \mathbb{Z})$. This leads us directly to a discussion of *Chern classes*.

Given a Kähler metric, we can define a matrix valued 2-form Θ of type $(1,1)$ by

$$\Theta_i^j = g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz_k \wedge d\bar{z}_l \,. \tag{3.68}$$

One defines the *Chern form*

$$c(M) = 1 + \sum_i c_i(M) = \det(\mathbb{1} + \frac{it}{2\pi}\Theta)|_{t=1} = (1 + t\phi_1(g) + t^2\phi_2(g) + \dots)|_{t=1} \tag{3.69}$$

which has the following properties (cf. e.g. [27, 12]):

- $d\phi_i(g) = 0$ and $[\phi_i] \in H^{i,i}(M, \mathbb{C}) \cap H^{2i}(M, \mathbb{R})$,
- $[\phi_i(g)]$ is independent of g,
- $c_i(M)$ is represented by $\phi_i(g)$.

$c_i(M)$ is the i^{th} Chern class of the manifold M. In these lectures we only need $c_1(M)$ which is expressed in terms of the Ricci form:

$$\phi_1(g) = \frac{i}{2\pi}\Theta_i^i = \frac{i}{2\pi} R_{k\bar{l}} dz^k \wedge d\bar{z}^l = \frac{1}{2\pi}\mathcal{R} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(g_{k\bar{l}}) \,.$$

For $c_1(M)$, the first two properties have been proven in (3.23) and (3.24). Moreover, if

$$dv = v dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$$

is any volume form on M, we can represent $c_1(M)$ by

$$c_1(M) = -\left[\frac{i}{2\pi}\partial\bar{\partial}\log(v)\right] \,. \tag{3.70}$$

This is so since $v = f\det(g)$ for a non-vanishing positive function f on M.

Example 3.4: Let $M = \mathbb{P}^n$, endowed with the Fubini-Study metric. We then have (cf. (3.29)) $\mathcal{R} = (n+1)\omega$, i.e. $c_1(\mathbb{P}^n) = \frac{1}{2\pi}(n+1)[\omega]$.

We say that $c_1(M) > 0\,(< 0)$ if $c_1(M)$ can be represented by a positive (negative) form. In local coordinates this means

$$\phi_1 = i\phi_{k\bar{l}} dz^k \wedge d\bar{z}^l\,, \tag{3.71}$$

where $\phi_{k\bar{l}}$ is a positive (negative) definite matrix. We say that $c_1(M) = 0$ if the first Chern class is cohomologous to zero. Clearly $c_1(\mathbb{P}^n) > 0$. Note that, e.g. a $c_1(M) > 0$ means that $\int_{\mathcal{C}} c_1 > 0$ for any curve $\mathcal{C}$ in M.

3.5 Calabi-Yau manifolds

We are now prepared to give a definition of a Calabi-Yau manifold:

A *Calabi-Yau manifold* is a compact Kähler manifold with vanishing first Chern class.

While it is obvious that any Ricci-flat Kähler manifold has vanishing first Chern class, the opposite is far from trivial. This problem was first considered by Calabi in a more general context. He asked the question whether any representative of $c_1(M)$ is the Ricci-form of some Kähler metric. (One can show that any two such representatives differ by a term of the form $\partial\bar{\partial}f$ where $f \in C^\infty(M,\mathbb{R})$. This is the content of the $\partial\bar{\partial}$-Lemma, cf. [12], 2.110.) Calabi also showed that if such a Kähler metric exists, then it must be unique. Yau provided the proof that such a metric always exists if M is compact.

The precise statement of Yau's theorem is: let M be a compact Kähler manifold, ω its Kähler form, $c_1(M)$ its first Chern class. Any closed real two-form of type (1,1) belonging to $2\pi c_1(M)$ is the Ricci form of one and only one Kähler metric in the class of ω.

For vanishing first Chern class, which is the case we are interested in, this means that given any Kähler metric g with associated Kähler form ω, one can always find a unique Ricci-flat Kähler metric g' with Kähler form ω' such that $[\omega] = [\omega']$, i.e. a Kähler manifold with $c_1(M) = 0$ admits a unique Ricci-flat Kähler form in each Kähler class.

Since the first Chern class is represented by the Ricci form and since the latter changes under change of metric by an exact form, i.e. $\mathcal{R}(g') = \mathcal{R}(g) + d\alpha$ (cf. (3.24)), vanishing of the first Chern class is necessary for having a Ricci-flat metric. This is the easy part of the theorem. To prove that this is also sufficient is the hard part. Yau's proof is an existence proof. In fact no Calabi-Yau metric has ever been constructed explicitly. In the non-compact case the situation in this respect is better; examples are the Eguchi-Hanson metrics, see e.g. [28], and the metric on the deformed and the resolved conifold [33]. They play a rôle in the resolution of singularities (orbifold and conifold singularities, respectively) which can occur in compact CY manifolds at special points in their moduli space.

The compact Kähler manifolds with zero first Chern class are thus precisely those which admit a Kähler metric with zero Ricci curvature, or equivalently, with restricted holonomy group contained in $SU(n)$. Following common practice we will talk about Calabi-Yau manifolds if the holonomy group is precisely $SU(n)$. This excludes tori and direct product spaces. We want to mention in passing that any compact Kähler manifold with $c_1(M) = c_2(M) = 0$ is flat, i.e. $M = \mathbb{C}^n/\Gamma$. This shows that while Ricci-flatness is characterized by the first Chern class, flatness is characterized by the second Chern class.

We should mention that the analysis that we sketched in the introduction, which led to considering Ricci-flat manifolds, was based on a perturbative string theory analysis which was further restricted to lowest order in α'. If one includes α'-corrections, both the beta-function equations and the supersymmetry transformations will be corrected and the Ricci-flatness condition is also modified. One finds the requirement $R_{i\bar{\jmath}} + \alpha'^3(R^4)_{i\bar{\jmath}} = 0$, where (R^4) is a certain tensor composed of four powers of the curvature. It has been shown that the α'-corrections to the Ricci-flat metric, which one has at lowest order, do not change the cohomology class. They are always of the form $\partial\bar{\partial}(...)$ and are thus cohomologically trivial [34].

One often defines Calabi-Yau manifolds as those compact complex Kähler manifolds with trivial canonical bundle. We now want to digress to explain the meaning of this statement and to demonstrate that it is equivalent to the definition given above. Chern classes can be defined for any complex vector bundle over M. By $c_i(M)$ as defined above we mean the Chern classes of the tangent bundle. Given a connection on the vector bundle, the Chern classes can be expressed by the curvature of the connection in the same way as for the tangent bundle with the hermitian connection.

Exercise 3.19: Show that $c_1(T^*M) = -c_1(TM)$.

A central property of Chern classes is that they do not depend on the choice of connection. They are topological cohomology classes in the base space of the vector bundle (see e.g. [25], p.90). An important class of vector bundles over a complex manifold are those with fibers of (complex) dimension one, the so called *line bundles* with fiber $\mathbb{C}$ (complex vector bundles of rank one). Holomorphic line bundles have holomorphic transition functions and a holomorphic section is given in terms of local holomorphic functions. Each holomorphic section defines a local holomorphic frame (which is, of course, one-dimensional for a line-bundle). One important and canonically defined line bundle is the *canonical line bundle* $K(M) = \overset{n}{\wedge} T^{*1,0}(M)$ whose sections are forms of type $(n,0)$, where $n = \dim_{\mathbb{C}}(M)$. It is straightforward to verify that $[\nabla_i, \nabla_{\bar{j}}]\omega_{i_1\dots,i_n} = -R_{i\bar{j}}\omega_{i_1\dots i_n}$, i.e. its curvature form is the negative of the Ricci form of the Kähler metric. This shows that $c_1(M) = -c_1(K(M))$ and if $c_1(M) = 0$ the first Chern class of the canonical bundle also vanishes. For a line bundle this means that it is trivial. Consequently there must exist a globally defined nowhere vanishing holomorphic n-form on M. One finds from (3.35) that on a compact Ricci-flat Kähler manifold any holomorphic p-form is covariantly constant. This means that the holonomy group $\mathcal{H}$ of a Calabi-Yau manifold is contained in $SU(n)$.

Example 3.5: In this example we consider complex hypersurfaces in $\mathbb{P}^n$ which are expressed as the zero set of a homogeneous polynomial. We already know that they are Kähler. We want to compute c_1 of the hypersurface as a function of the degree d of the polynomial and of n. From this we can read off the condition for the hypersurface to be a Calabi-Yau manifold. This can be done with the tools we have developed so far, even though more advanced and shorter derivations of the result can be found in the literature, see e.g. [23] or [12]. Later we will encounter another way to see that $d = n+1$ means $c_1(X) = 0$ by explicitly constructing the unique holomorphic n-form which, as we will see, must exist on a Calabi-Yau n-fold. The calculation is presented in Appendix C. The result we find there is

$$2\pi c_1(X) = (n+1-d)[\omega]. \tag{3.72}$$

It follows that the first Chern class $c_1(X)$ is positive, zero or negative according to $d < n+1$, $d = n+1$ and $d > n+1$, respectively.

We have thus found an easy way to construct Calabi-Yau manifolds. For one-folds, a cubic hypersurface in $\mathbb{P}^2$ is a 2-torus and for two-folds, a quartic hypersurface in $\mathbb{P}^3$ is a K3. If we are interested in three-folds, we have to choose the quintic hypersurface in $\mathbb{P}^4$. This is in fact the simplest example, which we will study further below.

The Calabi-Yau condition on the degree generalizes to the case of hypersurfaces in weighted projective spaces. Given a weighted projective space $\mathbb{P}^n[\boldsymbol{w}]$ and a hypersurface X specified by the vanishing locus of a quasi-homogeneous polynomial of degree d, we find

$$c_1(X) = 0 \quad \Leftrightarrow \quad d = \sum_{i=1}^{n} w_i \ . \tag{3.73}$$

The condition on the degrees and weights can also be easily written down for complete intersections in products of weighted projective spaces.

As we have discussed before, in the generic case the hypersurface will be singular. To get a smooth Calabi-Yau manifold one has to resolve the singularities in such a way that the canonical bundle remains trivial.

Example 3.6: An example of a CY_3 hypersurface in weighted projective space where no resolution is necessary is the sextic in $\mathbb{P}^4[1,1,1,1,2]$. The embedding space has only isolated singular points which are avoided by a generic hypersurface. On the other hand, the octic hypersurface in $\mathbb{P}^4[1,1,2,2,2]$ cannot avoid the singular $\mathbb{Z}_2$ surface of the embedding space and has thus itself a singular $\mathbb{Z}_2$ curve which must be 'repaired' in order to obtain a smooth CY manifold.

We should mention that the construction of the first Chern class that we present in Appendix C does not provide the Ricci-flat metric. In fact, the Ricci-flat metric is never the induced metric. As we have mentioned once before, a Ricci-flat Kähler metric on a *compact* Kähler manifold has never been constructed explicitly. Interesting examples of non-compact Ricci-flat Kähler manifolds, which are of potential interest for M-theory and the AdS/CFT correspondence, are the cotangent bundles of spheres of any dimension and the complex cotangent bundle on $\mathbb{P}^n$ for any n. The latter are hyper-Kähler manifolds, which are always Ricci-flat. For these manifolds Ricci-flat metrics are known explicitly. For instance, T^*S^3 is the deformed conifold.

Let us come back to the fact that a compact Kähler manifold with $SU(n)$ holonomy always possesses a nowhere vanishing covariantly constant $(n,0)$-form Ω, called a *complex volume form* which is in fact unique (up to multiplication by a constant). Locally it can always be written as

$$\Omega_{i_1\dots i_n} = f(z)\epsilon_{i_1\dots i_n} \tag{3.74}$$

with f a non-vanishing holomorphic function in a given coordinate patch and $\epsilon_{i_1\dots i_n} = \pm 1$. Before proving this we want to derive two simple corollaries:
(1) Ω is holomorphic. Indeed, $\bar\partial_{\bar\imath}\Omega_{j_1\dots j_n} = \nabla_{\bar\imath}\Omega_{j_1\dots j_n} = 0$, because Ω is covariantly constant.
(2) Ω is harmonic. To show this we still have to demonstrate $\bar\partial^*\Omega = 0$. But this obvious since $\bar\partial^* = -*\partial*$ and $*: A^{n,0}\to A^{n,0}$ and $\partial A^{n,0} = 0$.

A simple argument that Ω always exists is the following [35, 12]. Start at any point p in M and define $\Omega_p = dz^1\wedge\cdots\wedge dz^n$, where $\{z^i\}$ are local coordinates. Then parallel transport Ω to every other point on M. This is independent of the path taken, since when transported around a closed path (starting and ending at p), Ω is a singlet under $SU(n)$ and is thus unchanged. This defines Ω everywhere on M. Ω can also be constructed explicitly with the help of the covariantly constant spinor: $\Omega_{ijk} = \epsilon^{\mathrm{T}}\gamma_{ijk}\epsilon$. Here γ_{ijk} is the antisymmetrized product of three γ-matrices which satisfy $\{\gamma_i,\gamma_j\} = \{\gamma_{\bar\imath},\gamma_{\bar\jmath}\} = 0$, $\{\gamma_i,\gamma_{\bar\jmath}\} = 2g_{i\bar\jmath}$. The proof that Ω thus defined satisfies all the necessary properties is not difficult. It can be found in [28, 4].

We now show that Ω is essentially unique. Assume that given Ω there were a Ω' with the same properties. Then, since Ω is a form of the top degree, we must have $\Omega' = f\Omega$ where f is a non-singular function. Since we require $\bar\partial\Omega' = 0$, f must in fact be holomorphic. On a compact manifold this implies that f is constant.

Conversely, the existence of Ω implies $c_1 = 0$. Indeed, with (3.74), we can write the Ricci form as

$$\mathcal{R} = i\partial\bar\partial\log\det(g_{k\bar l}) = -i\partial\bar\partial\log\left(\Omega_{i_1\dots i_n}\bar\Omega_{\bar\jmath_1\dots\bar\jmath_n}g^{i_1\bar\jmath_1}\cdots g^{i_n\bar\jmath_n}\right) . \tag{3.75}$$

The argument of the logarithm is a globally defined function and the Ricci form is thus trivial in cohomology, implying $c_1 = 0$.

For hypersurfaces in weighted projective spaces one can explicitly construct Ω by extending the construction of holomorphic differentials on a Riemann surface (see e.g. [23]). Once constructed we know that Ω is essentially unique (up to a multiplicative constant on the hypersurface).

Consider first the torus defined as a hypersurface in $\mathbb{P}^2$ specified by the vanishing locus of a cubic polynomial, $f(x,y,z)=0$. This satisfies (3.72). The unique holomorphic differential (written in a patch with $z=1$) is $\omega=-dy/(\partial f/\partial x)=dx/(\partial f/\partial y)=dx/(2y)$. The first equality follows from $df=0$ along the hypersurface and the second equality if the hypersurface is defined by an equation of the form $f=zy^2-p(x,z)$, e.g. the Weierstrass and Legendre normal forms. An interesting observation is that ω can be represented as a residue: $\omega=\frac{1}{2\pi i}\int_\gamma \frac{dx\wedge dy}{f(x,y)}$. The integrand is a two-form in the embedding space with a first order pole on the hypersurface $f=0$ and the contour γ surrounds the hypersurface. Changing coordinates $(x,y)\to(x,f)$ and using $\frac{1}{2\pi i}\int_\gamma\frac{df}{f}=1$ we arrive at ω as given above.

The above construction of the holomorphic differential for a cubic hypersurface in $\mathbb{P}^2$ can be generalized to obtain the holomorphic three-form on a Calabi-Yau manifold realized as a hypersurface $p=0$ in weighted $\mathbb{P}^4[\boldsymbol{w}]$ [36, 37]. Concretely,

$$\Omega=\int_\gamma \frac{\mu}{p}\ , \tag{3.76}$$

where

$$\mu=\sum_{i=0}^{4}(-1)^i w_i z^i dz^0\wedge\ldots\wedge\widehat{dz^i}\wedge\ldots\wedge dz^5\ , \tag{3.77}$$

and the term under the $\widehat{\ }$ is omitted. The contour γ now surrounds the hypersurface $p=0$ inside the weighted projective space. Note that the numerator and the denominator in μ/p scale in the same way under (3.6). In the patch U_i where $z^i=\text{const}$, only one term in the sum survives. One can perform the integration by replacing one of the coordinates, say z^j, by p and using $\int_\gamma\frac{dp}{p}=2\pi i$. In this way one gets an expression for Ω directly on the embedded hypersurface. For instance in the patch U_0 one finds (no sum on (i,j,k) implied)

$$\Omega=\frac{w_0 z^0 dz^i\wedge dz^j\wedge dz^k}{\Delta_0^{ijk}}\ , \tag{3.78}$$

where $\Delta_0^{ijk}=\frac{\partial(z^i,z^j,z^k,p)}{\partial(z^1,z^2,z^3,z^4)}$. From our derivation it is clear that this representation of Ω is independent of the choice of $\{i,j,k\}\subset\{1,2,3,4\}$ and of the choice of coordinate patch. Furthermore, it is everywhere non-vanishing and well defined at every non-singular point of the hypersurface. A direct verification of these properties can be found in [38, 4].

The existence of a holomorphic n-form then means that the holonomy group $\mathcal{H}$ (and not just $\mathcal{H}_0$) is contained in $SU(n)$.

Let us now complete the discussion of Hodge numbers of Calabi-Yau manifolds. We have just established the existence of a unique harmonic $(n,0)$-form, Ω, and thus

$$h^{n,0}=h^{0,n}=1\,. \tag{3.79}$$

With the help of Ω we can establish one further relation between the Hodge numbers. Given a holomorphic and hence harmonic $(p,0)$-form, we can, via contraction with Ω, construct a $(0,n-p)$-form, which can be shown to be again harmonic, as follows. Given

$$\alpha=\alpha_{i_1\ldots i_p}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\ ,\qquad \bar\partial\alpha=0\,, \tag{3.80}$$

α being (Δ_∂)-harmonic means

$$\begin{aligned}\partial\,\alpha=0 &\qquad\Leftrightarrow\qquad \nabla_{[j_i}\alpha_{j_2\ldots j_{p+1}]}=0\ ,\\ \partial^*\alpha=0 &\qquad\Leftrightarrow\qquad \nabla^{i_1}\alpha_{i_1\ldots i_p}=0\ .\end{aligned} \tag{3.81}$$

We then define the $(0,n-p)$-form

$$\beta_{\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n}=\frac{1}{p!}\bar\Omega_{\bar{\jmath}_1\cdots\bar{\jmath}_n}\alpha^{\bar{\jmath}_1\cdots\bar{\jmath}_p}\,. \tag{3.82}$$

This can be inverted to give (use (3.74))

$$\alpha^{\bar{\jmath}_1\cdots\bar{\jmath}_p} = \frac{1}{\|\Omega\|^2}\Omega^{\bar{\jmath}_1\cdots\bar{\jmath}_p\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n}\beta_{\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n}\,, \tag{3.83}$$

where we have defined

$$\|\Omega\|^2 = \frac{1}{n!}\Omega_{i_1\ldots i_n}\Omega^{i_1\ldots i_n}\,. \tag{3.84}$$

From this we derive

$$\nabla^{\bar{\jmath}_{p+1}}\beta_{\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n} = \frac{1}{p!}\bar{\Omega}_{\bar{\jmath}_1\cdots\bar{\jmath}_n}\nabla^{\bar{\jmath}_{p+1}}\alpha^{\bar{\jmath}_1\cdots\bar{\jmath}_p} = 0\,, \tag{3.85}$$

using $(3.81)_1$. Similarly

$$\nabla_{\bar{\jmath}_1}\alpha^{\bar{\jmath}_1\cdots\bar{\jmath}_p} = \frac{1}{\|\Omega\|^2}\Omega^{\bar{\jmath}_1\cdots\bar{\jmath}_p\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n}\nabla_{\bar{\jmath}_1}\beta_{\bar{\jmath}_{p+1}\cdots\bar{\jmath}_n} = 0 \tag{3.86}$$

by virtue of $(3.81)_2$. It follows that β is also harmonic.

We have thus shown the following relation between Hodge numbers

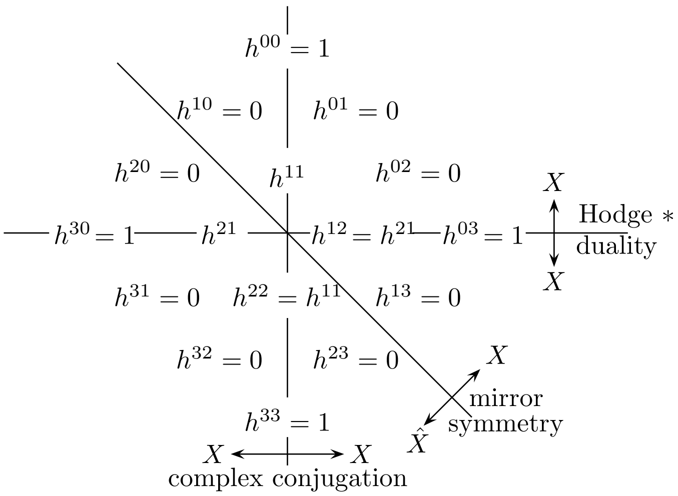
$$h^{p,0} = h^{0,n-p} = h^{n-p,0}\,. \tag{3.87}$$

Let us finally look at $h^{p,0}$. For this we need the Laplacian on p-forms. Specifying (3.35) for a harmonic $(p,0)$ form on a Ricci-flat Kähler manifold where $R_{i\bar{\jmath}} = R_{ij\bar{k}\bar{l}} \equiv 0$, we find $\nabla^\nu\nabla_\nu\omega_{i_1\cdots i_p} = 0$. On a compact manifold this means that ω is parallel, i.e. $\nabla_j\omega_{i_1\cdots i_p} = 0$, $\bar{\partial}_{\bar{\jmath}}\omega_{i_1\cdots i_p} = 0$, the latter equality already being a consequence of harmonicity. But this means that ω transforms as a singlet under the holonomy group. We now assume that the holonomy group is exactly $SU(n)$, i.e. not a proper subgroup of it.[9] Since $\omega_{i_1\cdots i_p}$ transforms in the $\wedge^p\boldsymbol{n}$ of $SU(n)$, the singlet only appears in the decomposition if $p=0$ or $p=n$. We thus learn that on Calabi-Yau manifolds with holonomy group $SU(n)$

$$h^{p,0} = 0 \qquad \text{for} \qquad 0 < p < n\,. \tag{3.88}$$

Exercise 3.20: Show that $h^{1,0}(M) = 0$ implies that there are no continuous isometries on M.

If we collect the results on the Hodge numbers of Calabi-Yau manifolds for the case $n=3$, we find that the only independent Hodge numbers are $h^{1,1} \geq 1$ and $h^{2,1} \geq 0$ and the Hodge diamond for Calabi-Yau three-folds is



(3.89)

[9] In Chapter 4 we discuss orbifolds with discrete holonomy groups. There the condition will be that it is not contained in any continuous subgroup of $SU(n)$.

The Euler number of a Calabi-Yau three-fold is then (cf. (3.64))

$$\chi(M_3) = 2(h^{1,1} - h^{1,2})\,. \tag{3.90}$$

In higher dimensions there are more independent Hodge numbers, but this will not be covered here. For the case of CY four-folds, see [39]. The significance of $h^{1,1}$ and $h^{2,1}$ for Calabi-Yau three-folds will be explained in sect. 3.6.

In (3.89) we have indicated operations which relate Hodge numbers to each other. In addition to complex conjugation (3.65) and the Hodge $*$-operation (3.66), which act on the Hodge numbers of a given CY manifold, we have also shown the action of mirror symmetry: given a CY manifold X, there exists a mirror manifold $\hat{X}$ such that $h^{p,q}(X) = h^{3-p,q}(\hat{X})$. This in particular means that the two non-trivial Hodge numbers $h^{1,1}$ and $h^{2,1}$ are interchanged between X and $\hat{X}$ and that $\chi(X) = -\chi(\hat{X})$. Within the class of Calabi-Yau manifolds constructed as hypersurfaces in toric varieties (which we have not discussed) they manifestly come in mirror pairs [40]. So-called rigid manifolds, for which $h^{2,1}(X) = 0$, and consequently $h^{1,1}(\hat{X}) = 0$, are discussed in [41]. The Z-manifold described in example 4.4 is rigid.

As we have already mentioned at the end of Chapter 2, mirror symmetry is a much stronger statement than the mere existence of mirror pairs of CY manifolds. Its far-reaching consequences for both string theory and algebraic geometry are thoroughly covered in [18].

3.6 Calabi-Yau moduli space

In this section we will only treat three dimensional Calabi-Yau manifolds. References are [4, 38, 42, 21, 43]. The generalization to higher dimensions of most the issues discussed here is straightforward. The two-dimensional case (K3) is described in [44] in great detail.

In view of Yau's theorem, the parameter space of CY manifolds is that of Ricci-flat Kähler metrics. We thus ask the following question: given a Ricci-flat Riemannian metric $g_{\mu\nu}$ on a manifold M, what are the allowed infinitesimal variations $g_{\mu\nu} + \delta g_{\mu\nu}$ such that

$$R_{\mu\nu}(g) = 0 \quad \Rightarrow \quad R_{\mu\nu}(g + \delta g) = 0 \qquad ? \tag{3.91}$$

Clearly, if g is a Ricci-flat metric, then so is any metric which is related to g by a diffeomorphism (coordinate transformation). We are not interested in those δg which are generated by a change of coordinates. To eliminate them we have to fix the diffeomorphism invariance and impose a *coordinate condition*. This is analogous to fixing a gauge in electromagnetism. The appropriate choice is to demand that $\nabla^\mu \delta g_{\mu\nu} = 0$ (see e.g. [12], 4.62). Any $\delta g_{\mu\nu}$ which satisfies this condition also satisfies $\int_M \sqrt{g}\delta g^{\mu\nu}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) d^d x = 0$, and is thus orthogonal to any change of the metric induced by a diffeomorphism generated by the vector field ξ_μ. Then, expanding (3.91) to first order in δg and using $R_{\mu\nu}(g) = 0$ and the coordinate condition, one finds

$$\nabla^\rho\nabla_\rho \delta g_{\mu\nu} - 2R_\mu{}^\rho{}_\nu{}^\sigma \delta g_{\rho\sigma} = 0\ . \tag{3.92}$$

One needs to use that M is compact to eliminate a term $\nabla_\mu\nabla_\nu \mathrm{tr}(\delta g)$.

Exercise 3.21: Derive (3.92). Useful expansions of the curvature can be found in [45].

We now want to analyze (3.92) if (M, g) is a Kähler manifold. Given the index structure of the metric and the Riemann tensor on Kähler manifolds, one immediately finds that the conditions imposed on the components $\delta g_{i\bar{j}}$ and δg_{ij} decouple and can thus be studied separately. This is what we now do in turn.

(1) $\delta g_{i\bar{j}}$: With the help of (3.35) it is easy to see that the condition (3.92), which now reads $\nabla^\mu\nabla_\mu \delta g_{i\bar{j}} - 2R_i{}^k{}_{\bar{j}}{}^{\bar{l}} \delta g_{k\bar{l}} = 0$, $\mu{=}(k, \bar{k})$, is equivalent to $(\Delta\delta g)_{i\bar{j}} = 0$. Here we view $\delta g_{i\bar{j}}$ as the components of a $(1,1)$-form. We see that harmonic $(1,1)$-forms correspond to the metric variations of the form $\delta g_{i\bar{j}}$ and to cohomologically non-trivial changes of the Kähler form. Of course, we already knew from Yau's theorem

that for any $[\omega+\delta\omega]$ there is again a Ricci-flat Kähler metric. Expanding $\delta g_{i\bar{\jmath}}$ in a basis of real $(1,1)$-forms, which we will denote by b^α, $\alpha = 1,\ldots,h^{1,1}$, we obtain the following general form of the deformations of the Kähler structure of the Ricci flat metric:

$$\delta g_{i\bar{\jmath}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{t}^\alpha b^\alpha_{i\bar{\jmath}}\,, \qquad \tilde{t}^\alpha \in \mathbb{R}\,. \tag{3.93}$$

Using (3.56) one may check that these δg satisfy the coordinate condition.

For $g+\delta g$ to be a Kähler metric, the *Kähler moduli* $\tilde{t}^\alpha$ have to be chosen such that the deformed metric is still positive definite. Positive definiteness of a metric g with associated Kähler form ω is equivalent to the condition

$$\int_C \omega > 0\,, \qquad \int_S \omega^2 > 0\,, \qquad \int_M \omega^3 > 0 \tag{3.94}$$

for all curves C and surfaces S on the Calabi-Yau manifold M. The subset in $\mathbb{R}^{h^{1,1}}$ spanned by the parameters $\tilde{t}^\alpha$ such that (3.94) is satisfied, is called the *Kähler cone*.

Exercise 3.22: Verify that this is indeed a cone.

(2) δg_{ij} : Now (3.92) reads $\nabla^\mu\nabla_\mu\delta g_{ij} - 2R_i{}^k{}_j{}^l\delta g_{kl} = 0$. With little work this can be shown to be equivalent to

$$\Delta_{\bar{\partial}}\delta g^i = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\delta g^i = 0 \tag{3.95}$$

where

$$\delta g^i = \delta g^i_{\bar{\jmath}} d\bar{z}^{\bar{\jmath}}\,, \qquad \delta g^i_{\bar{\jmath}} = g^{i\bar{k}}\delta g_{\bar{k}\bar{\jmath}} \tag{3.96}$$

is a $(0,1)$-form with values in $T^{1,0}(M)$. We conclude that (3.95) implies that $\delta g^i \in H^{(0,1)}_{\bar{\partial}}(M,T^{1,0})$. Again one may verify that these deformations of the metric satisfy the coordinate condition.

Exercise 3.23: Fill in the steps of the above argument.

What is the significance of these metric deformations? For the new metric to be again Kähler, there must be a coordinate system in which it has only mixed components. Since holomorphic coordinate transformations do not change the type of index, it is clear that δg_{ij} can only be removed by a non-holomorphic transformation. But this means that the new metric is Kähler with respect to a different complex structure compared to the original metric. (Of course, this new metric cannot be obtained from the original undeformed metric by a diffeomorphism as they have been fixed by the coordinate condition. So while removing δg_{ij}, the non-holomorphic change of coordinates generates a $\delta g_{i\bar{\jmath}}$).

With the help of the unique holomorphic $(3,0)$ form we can now define an isomorphism between $H^{0,1}_{\bar{\partial}}(M,T^{1,0})$ and $H^{2,1}_{\bar{\partial}}(M)$ by defining the complex (2,1)-forms

$$\Omega_{ijk}\delta g^k_{\bar{l}} dz^i \wedge dz^j \wedge \bar{z}^{\bar{l}}\,. \tag{3.97}$$

which are harmonic if (3.92) is satisfied. These complex structure deformations can be expanded in a basis $b^a_{ij\bar{k}}$, $a = 1,\ldots,h^{2,1}$, of harmonic $(2,1)$-forms:

$$\Omega_{ijk}\delta g^k_{\bar{l}} = \sum_{a=1}^{h^{2,1}} t^a b^a_{ij\bar{l}} \tag{3.98}$$

where the complex parameters t^a are called *complex structure moduli.* [10]

[10] Our discussion of complex structure moduli is not complete. We have only considered the linearized deformation equation. It still needs to be shown that they can be integrated to finite deformations. That this is indeed the case for Calabi-Yau manifolds has been proven by Tian [46] and by Todorov [47]. For a general complex manifold the number of complex structure deformations is less than $h^{2,1}$.

If we were geometers we would only be interested in the deformations of the metric and the number of real deformation parameters (moduli) would be $h^{1,1} + 2h^{1,2}$. However, in string theory compactified on Calabi-Yau manifolds we have additional massless scalar degrees of freedom from the internal components of the antisymmetric tensor field in the (NS,NS) sector of the type II string. Its equations of motion in the gauge $d^*B = 0$ are $\Delta B = 0$, i.e. excitations of the B-field above the background where it vanishes are harmonic two-forms on the Calabi-Yau manifold. We can now combine these with the Kähler deformations of the metric and form

$$(i\delta g_{i\bar{\jmath}} + \delta B_{i\bar{\jmath}})dz^i \wedge d\bar{z}^{\bar{\jmath}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{t}^\alpha b^\alpha \tag{3.99}$$

where the parameters $\tilde{t}^\alpha$ are now complex, their imaginary part still restricted by the condition discussed before. This is referred to as the *complexification of the Kähler cone*.

To summarize, there is a moduli space associated with the different Kähler and complex structures which are compatible with the Calabi-Yau condition. The former are parametrized by $H^{1,1}_{\bar{\partial}}(M)$ and the latter by of $H^{0,1}_{\bar{\partial}}(M, T^{1,0}) \simeq H^{2,1}_{\bar{\partial}}(M)$. The moduli space of Ricci-flat Kähler metrics is parametrized by the harmonic representatives of these cohomology groups.

Let us now exemplify this discussion by the quintic in $\mathbb{P}^4$. Here we have $h^{1,1} = 1$: this is simply the Kähler form induced from the ambient space $\mathbb{P}^4$. (The metric induced from the Fubini-study metric is not the Ricci-flat one.) As shown in [26] and by more elementary means in [38], the complex structure parameters appear as coefficients in the most general quintic polynomial. One easily finds that there are 126 coefficients. However, polynomials which are related by a linear change of the homogeneous coordinates of $\mathbb{P}^4$ should not be counted as different. These are parametrized by $\dim_{\mathbb{C}}(GL(5,\mathbb{C})) = 25$ coefficients. We therefore conclude that there are 101 complex structure moduli on the quintic hypersurface, i.e. $h^{2,1} = 101$. For special values of these coefficients the hypersurface is singular, i.e. there are solutions of $p = dp = 0$. With (3.90) we find that the Euler number of the quintic is -200.

The situation for hypersurfaces in weighted projective spaces is more complicated. If the hypersurface passes through the singular loci of the embedding space, they have to be 'repaired'. Care has to be taken that in doing this the Calabi-Yau condition $c_1 = 0$ is maintained. This introduces additional elements in the cohomology, so that in general $h^{1,1} > 1$. Also $h^{2,1}$ can no longer be counted as the number of coefficients in the defining polynomial: this counting falls short of the actual number of complex structure moduli. There are methods to compute the Hodge numbers of these manifolds. The most systematic and general one is by viewing them as hypersurfaces in toric varieties [40].

We will not address questions of global properties of the moduli space of string compactifications on Calabi-Yau manifolds, except for mentioning a few aspects. Mirror symmetry, which connects topologically distinct manifolds, is certainly relevant. Another issue is that of transitions among topologically different manifolds, the prime example being the conifold transition [33]. While one encounters singular geometries in the process, string theory is well behaved and the transition is smooth. Indeed, it has been speculated that the moduli space of all Calabi-Yau compactifications is smoothly connected [48].

3.7 Compactification of Type II supergravities on a CY three-fold

Now that we know the meaning of the Hodge numbers $h^{1,1}$ and $h^{2,1}$, we can, following our general discussion in sect. 2.3, examine the relevance of the existence of harmonic forms on Calabi-Yau manifolds for the massless spectrum of the compactified theory. We will consider the two ten-dimensional type II supergravities that are the field theory limits of type II strings. The discussion is thus also relevant for string compactification, as long as the restriction to the massless modes is justified, i.e. for energies $E^2\alpha' \ll 1$. However, there are string effects which are absent in field theory compactifications, such as topological non-trivial embeddings of the string world-sheet into the CY manifold. These stringy effects (world-sheet instantons) which are non-perturbative in α', have an action which scales as R^2/α', where R is the typical size of the manifold. They are suppressed as $e^{-S_{\rm inst}} \sim e^{-R^2/\alpha'}$ and are small for a large internal manifold but relevant for $R \sim \sqrt{\alpha'}$.

Type IIA supergravity is a non-chiral $\mathcal{N}=2$ theory with just a gravity multiplet whose field content is:

$$\mathcal{G}_{\rm IIA}(10) = \{G_{MN}, \psi_M^{(+)}, \psi_M^{(-)}, \psi^{(+)}, \psi^{(-)}, B_{MN}, A_{MNP}, V_M, \phi\}. \tag{3.100}$$

These fields correspond to the massless states of the type IIA string. The fermionic fields arise in the two Neveu-Schwarz-Ramond sectors, i.e. (NS,R) plus (R,NS), they are the two Majorana-Weyl gravitini of opposite chirality $\psi_M^{(\pm)}$, $M, N = 0, \cdots, 9$, and the two Majorana-Weyl dilatini $\psi^{(\pm)}$. The metric G_{MN}, the antisymmetric tensor B_{MN} and the dilaton ϕ come from the (NS,NS) sector. The remaining bosonic fields, the vector V_M and the 3-index antisymmetric tensor A_{MNP}, appear in the (R,R) sector.

Exercise 3.24: Show that (3.100) results upon circle compactification of the fields $\{G_{MN}, \psi_M, A_{MNP}\}$, with ψ_M Majorana. This is the matter content of $D=11$ supergravity which is the low-energy limit of M-theory.

Type IIB supergravity has also $\mathcal{N}=2$ supersymmetry but it is chiral, i.e. the two gravitini have the same chirality. The gravity multiplet has content:

$$\mathcal{G}_{\rm IIB}(10) = \{G_{MN}, \psi_M^{(+)}, \tilde{\psi}_M^{(+)}, \psi^{(+)}, \tilde{\psi}^{(+)} B_{MN}, \tilde{B}_{MN}, A_{MNPQ}, \phi, a\}. \tag{3.101}$$

Now the bosonic fields from the (R,R) sector are the axion a, $\tilde{B}_{MN}$ and A_{MNPQ} which is completely antisymmetric and has self-dual field strength.

It is known that type IIA and type IIB strings compactified on a circle are related by T-duality [49]. Therefore, whenever the internal manifold contains a circle, type IIA and type IIB give T-dual theories that clearly must have the same supersymmetric structure. In particular, compactification on T^4 gives maximal (2,2) supersymmetry in $d=6$, compactification on T^6 gives maximal $\mathcal{N}=8$ supersymmetry in $d=4$ and compactification on $\mathrm{K3} \times \mathrm{T}^2$ gives $d=4$, $\mathcal{N}=4$ supersymmetry with 22 $U(1)$ vector multiplets. Below we examine compactification on CY_3 in some more detail. Our purpose is to determine the resulting massless fields by looking at the zero modes of the ten-dimensional multiplets given above. In the lower dimensions we will obtain a theory with a number of supersymmetries that depends on the internal manifold. Clearly, the zero modes must organize into appropriate multiplets whose structure is known beforehand.

Compactification of type IIA supergravity on a CY_3 was considered first in [50] and to greater extent in [51]. The resulting theory in $d = 4$ has $\mathcal{N} = 2$ supersymmetry. The massless fields belong to the gravity multiplet plus hypermultiplets and vector multiplets, which are the three possible irreducible representations with spins less or equal to two, cf. (4.65). To describe how the massless fields arise we split the ten-dimensional indices in a $SU(3)$ covariant way, $M = (\mu, i, \bar{\imath})$[11] and then use the known results for the number of harmonic (p, q) forms on the Calabi-Yau manifold. The zero modes of $G_{\mu\nu}$, $\psi_\mu^{(+)}$, $\psi_\mu^{(-)}$ and the graviphoton V_μ form the gravity multiplet. Both $\psi_\mu^{(\pm)}$ have an expansion of the form (2.19) so that we obtain two Majorana gravitini in four dimensions. For the remaining fields and components it is simpler to analyze the bosonic states. The fermions are most easily determined via $\mathcal{N}=2$ space-time supersymmetry and the known field content of the various multiplets. Of course they can also be obtained by a zero mode analysis. Altogether one finds for the bosons

$$A_\mu^\alpha,\ t^a,\ \tilde{t}^\alpha,\ C^a,\ S,\ C\ , \tag{3.102}$$

[11] From now on we only use indices $(i, j, \ldots, \bar{\imath}, \bar{\jmath}, \ldots)$ for the internal space and μ for the four uncompactified space-time dimensions.

where A_μ^α arises from $A_{\mu i\bar{\jmath}}$ and the remaining fields are all complex scalars as follows. The $\tilde{t}^\alpha$ correspond to $G_{i\bar{\jmath}}$ and $B_{i\bar{\jmath}}$[12], the t^a to G_{ij}, C^a to the $A_{ij\bar{k}}$ modes, S to ϕ and $B_{\mu\nu}$ (which can be dualized to a pseudoscalar) and C to the A_{ijk} mode. We now group these fields into supermultiplets. A_μ^α and $\tilde{t}^\alpha$ combine to $h^{1,1}$ vector multiplets, whereas t^a and C^a to $h^{2,1}$ hypermultiplets. The two complex scalars S and C form an additional hypermultiplet, so there are $(h^{2,1}+1)$ hypermultiplets.

In the type IIB compactification the gravity multiplet is formed by the zero modes of $G_{\mu\nu}$, $\psi_\mu^{(+)}$, $\tilde{\psi}_\mu^{(+)}$ and $A_{\mu ijk}$. From the rest of the fields we obtain

$$A_\mu^a,\ t^a,\ \tilde{t}^\alpha,\ C^\alpha,\ S,\ C\ . \tag{3.103}$$

Here the fields A_μ^a arise from $A_{\mu ij\bar{k}}$ and t^a from G_{ij}; $(\tilde{t}^\alpha, C^\alpha)$ correspond to $G_{i\bar{\jmath}}$, $B_{i\bar{\jmath}}$, $\tilde{B}_{i\bar{\jmath}}$ and $A_{\mu\nu i\bar{\jmath}}$; (S,C) to ϕ, a, $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$. Altogether the fields combine to $(h^{1,1}+1)$ hypermultiplets and $h^{2,1}$ vector multiplets. Notice that this is the same result as in the type IIA case upon exchanging $h^{1,1}$ and $h^{1,2}$. Indeed, it has been shown that compactification of type IIB strings on a CY three-fold X gives the same 4-dimensional theory that appears upon compactification of type IIA strings on the mirror $\hat{X}$ [50, 52].

The moduli of the Calabi-Yau manifold give rise to neutral massless scalars that will appear in the low-energy effective action of the string theory. Supersymmetry imposes stringent restrictions on the action and consequently on the geometry of the moduli spaces. In particular, the moduli fields have no potential and hence their vevs are free parameters. Moreover, in the kinetic terms scalars in vector multiplets do not mix with scalars in hypermultiplets. In fact, the interaction of vector multiplets and hypermultiplets consistent with $\mathcal{N}=2$ supergravity is a non-linear σ-model with a target-space geometry which is locally of the form [53, 54, 55]

$$\mathcal{M}_{\rm SK}\times\mathcal{Q} \tag{3.104}$$

where $\mathcal{M}_{\rm SK}$ is a (special) Kähler manifold (to be defined later) for the vector multiplets [53] and $\mathcal{Q}$ a quaternionic manifold for the hypermultiplets [53, 54, 55].[13] The manifolds $\mathcal{M}_{\rm SK}$ and $\mathcal{Q}$ are parametrized by the scalar fields inside the vector and hypermultiplets, respectively. The product structure is only respected for the gauge-neutral part of the theory. Nonabelian gauge symmetries and charged fields appear if we take non-perturbative effects into account, e.g. by wrapping branes around appropriate cycles. But this will not be considered in these lectures.

For the perturbative type IIA and IIB theories we thus have

$$\begin{aligned}\mathcal{M}^A &= \mathcal{M}^A_{h^{1,1}}\times\mathcal{Q}^A_{h^{2,1}+1}\,,\\ \mathcal{M}^B &= \mathcal{M}^B_{h^{2,1}}\times\mathcal{Q}^B_{h^{1,1}+1}\,.\end{aligned} \tag{3.105}$$

The indices give the complex and quaternionic dimensions, respectively. It is worth mentioning that while $\mathcal{M}_{\rm SK}$ contains only moduli fields, $\mathcal{Q}$ is obtained by combining moduli scalars with non-moduli scalars which, in string theory, come from the (R,R) sector of the left-right superconformal algebra.

The quaternionic dimension of the hypermultiplet moduli spaces is always ≥ 1. In both type II theories, there is at least the *universal hypermultiplet* with scalars (S,C). Its component fields are not related to the cohomology of a Calabi-Yau manifold. Most importantly, it contains the dilaton ϕ which organizes the string perturbation theory. This means that the hypermultiplet moduli space receives (perturbative and non-perturbative) stringy corrections in type IIA and IIB. In contrast to this, the vector multiplet moduli space is exact at string tree level. In types IIB and IIA this concerns the complex structure moduli and Kähler moduli, respectively. The metric of the Kähler moduli space of type IIA receives a perturbative correction at order $(\alpha'/R^2)^3$ [56] and non-perturbative corrections, powers of $e^{-R^2/\alpha'}$, from world-sheet instantons, i.e. topologically non-trivial embeddings of the world-sheet into the

[12] Supersymmetry thus requires the complexification of the Kähler cone.

[13] A quaternionic manifold is a complex manifold of dimension $4m$ and holonomy group $Sp(1)\times Sp(m)$.

Calabi-Yau manifold. In contrast, the metric of the complex structure moduli space of type IIB is exact at both, string and world-sheet σ-model, tree level. It is thus determined by classical geometry. The vector multiplet moduli space of the type IIA theory, on the other hand, is not determined by classical geometry, but rather by 'string geometry'. The string effects are suppressed at large distances, i.e. when the Calabi-Yau manifold on which we compactify becomes large. At small distances, of the order of the string scale $l_s = 1/\sqrt{\alpha'}$, the intuition derived from classical geometry fails.

It thus looks hopeless to compute the vector multiplet moduli space of the type IIA theory. Here mirror symmetry comes to rescue as was first shown, for the case of the quintic in $\mathbb{P}^4$, in [56]. It relates, via the mirror map, the vector multiplet moduli space of the type IIA theory on X to the vector multiplet moduli space of the type IIB theory on the mirror $\hat{X}$. Thus, with the help of mirror symmetry the structure of the vector multiplet moduli space of both type II theories is understood, as long as the conditions which lead to (3.104) are met. Due to lack of space we have to refer to the literature for any details [21, 32, 18].

One obtains the moduli space of the heterotic string by setting the (R,R) fields to zero. This gives

$$\mathcal{M}^{\rm het} = \frac{SU(1,1)}{U(1)} \times \mathcal{M}_{h^{1,1}} \times \mathcal{M}_{h^{2,1}} \tag{3.106}$$

where the second and third factors are special-Kähler manifolds. (3.106), which was derived in [55, 50, 48, 57, 58], is only valid at string tree level. The loop corrections which destroy the product structure have been computed in [59].

We will now briefly explain the notion of a special Kähler manifold which arises in the construction of $\mathcal{N} = 2$ supersymmetric couplings of vector multiplets to supergravity. It was found that the entire Lagrangian can be locally encoded in a holomorphic function $F(t)$, where t^a are (so-called special) coordinates on the space spanned by the scalar fields inside the vector multiplets. For instance, in type IIB compactification on a CY$_3$, this is the complex structure moduli space and $a = 1, \ldots, h^{2,1}$. Supersymmetry requires that this space is Kähler and furthermore, that its Kähler potential can be expressed through F via

$$\begin{aligned} K &= -\ln Y \\ Y &= 2(F - F) - (t^a - \bar{t}^a)(F_a + \bar{F}_a) \end{aligned} \tag{3.107}$$

where $F_a = \partial_a F$. For this reason F is called the (holomorphic) *prepotential*. If we introduce projective coordinates z via $t^a = z^a/z^0$ and define $\mathcal{F}(z) = (z^0)^2 F(t)$ we find that the Kähler potential (3.107) can be written, up to a Kähler transformation, as

$$K = \ln\left(\bar{z}^a \mathcal{F}_a - z^a \bar{\mathcal{F}}_a\right) \tag{3.108}$$

where now $a = 0, \ldots, h^{2,1}$, and $\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial z^a}$. Supersymmetry requires furthermore that $\mathcal{F}$ is a homogeneous function of degree two.

We will now show how these features are encoded in the CY geometry. We begin by introducing a basis of $H^3(X, \mathbb{Z})$ with generators α_a and β^b $(a, b = 0, \ldots, h^{2,1}(X))$ which are (Poincaré) dual to a canonical homology basis (B_a, A^b) of $H_3(X, \mathbb{Z})$ with intersection numbers $A^a \cdot A^b = B_a \cdot B_b = 0$, $A^a \cdot B_b = \delta^a_b$. Then

$$\int_{A^b} \alpha_a = \int_X \alpha_a \wedge \beta^b = -\int_{B_a} \beta^b = \delta^b_a \tag{3.109}$$

All other pairings vanish. This basis is unique up to $Sp(2h^{2,1} + 2, \mathbb{Z})$ transformations.

Following [60, 61], one can show that the A-periods of the holomorphic (3,0)-form Ω, i.e. $z^a = \int_{A^a} \Omega$ are local projective coordinates on the complex structure moduli space. We then have for the B-periods $\mathcal{F}_a = \int_{B_a} \Omega = \mathcal{F}_a(z)$. Note that $\Omega = z^a \alpha_a - \mathcal{F}_a \beta^a$. Furthermore, under a change of complex structure

Ω, which was pure $(3,0)$ to start with, becomes a mixture of $(3,0)$ and $(2,1)$ (because dz in the old complex structure becomes a linear combination of dz and $d\bar{z}$ w.r.t. to the new complex structure): $\frac{\partial}{\partial z^a}\Omega \in H^{(3,0)} \oplus H^{(2,1)}$. In fact [27, 38, 42] $\frac{\partial \Omega}{\partial z^a} = k_a \Omega + b_a$ where $b_a \in H^{(2,1)}(X)$ and k_a is a function of the moduli but independent of the coordinates on X (since Ω is unique). One immediate consequence is that $\int \Omega \wedge \frac{\partial \Omega}{\partial z^a} = 0$. Inserting the expansion for Ω in the α_a, β^a basis into this equation, one finds $\mathcal{F}_a = \frac{1}{2}\frac{\partial}{\partial z^a}(z^b \mathcal{F}_b)$, or $\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial z^a}$ with $\mathcal{F} = \frac{1}{2} z^a \mathcal{F}_a$, $\mathcal{F}(\lambda z) = \lambda^2 \mathcal{F}(z)$. We thus identify z^a with the special coordinates of supergravity and $\mathcal{F}$ with the prepotential. It is easy to verify that the Kähler potential in the form (3.108) can be written as $K = -\ln \int \Omega \wedge \bar{\Omega}$. In fact, $\mathcal{F}$ can be explicitly computed for the complex structure moduli space of type IIB theory in terms of the periods of the holomorphic three-form. This is a calculation in classical geometry. The Kähler moduli space of type IIA theory is also characterized by a prepotential. However its direct calculation is very difficult since it receives contributions from world-sheet instantons. Mirror symmetry relates $\mathcal{F}^{\text{Kahler}}(X)$ to the prepotential of the complex structure moduli space on the mirror manifold $\mathcal{F}^{\text{complex}}(\hat{X})$ which can be computed and mapped, via the mirror map, to $\mathcal{F}^{\text{Kahler}}(X)$ (see e.g. [21] for a review).

In supergravity and superstring compactifications many other properties of special Kähler manifolds are relevant e.g. in the explicit construction of the mirror map, the computation of Yukawa couplings in heterotic compactifications, etc. All these details can be found in the cited references. Ref.[62] discusses some subtle issues involving the existence of a prepotential (but see also [63] for their irrelevance in string compactification on CY manifolds once world-sheet instanton effects are included).

While, as we have seen, a great deal is known about the (local) geometry of the vector multiplet moduli space, the question about the structure of the hypermultiplet moduli space, except that it is a quaternionic manifold, is still largely unanswered and a subject of ongoing research. The difficulty comes, of course, from the fact that it receives perturbative and non-perturbative quantum corrections. Some partial results have been obtained e.g. in [64, 65].

4 Strings on orbifolds

We now want to consider string compactifications in which the internal space belongs to a class of toroidal orbifolds that are analogous to Calabi-Yau spaces in that their holonomy group is contained in $SU(n)$ and therefore the theory in the lower dimensions has unbroken supersymmetry. Even though these orbifolds are singular, we will see that string propagation is perfectly consistent provided that twisted sectors are included. Moreover, since toroidal orbifolds are flat except at fixed points, the theory is exactly solvable. Indeed, the fields on the world-sheet satisfy free equations of motion with appropriate boundary conditions.

In this section we will first discuss some basic properties of orbifolds. We next describe in some detail the compactification of strings on orbifolds, introducing in the process the important concepts of partition function and modular invariance. Finally the general results are applied to type II theories. In appendix C we collect some useful results about the partition function of $\mathrm{T}^6/\mathbb{Z}_N$ orbifolds.

The standard references for strings on orbifolds are the original papers [3]. A concise review that also discusses conformal field theory aspects is [66].

4.1 Orbifold geometry

In general, an orbifold $\mathcal{O}$ is obtained by taking the quotient of a manifold $\mathcal{M}$ by the action of a discrete group G that preserves the metric of $\mathcal{M}$. This means:

$$\mathcal{O} = \mathcal{M}/G \ . \tag{4.1}$$

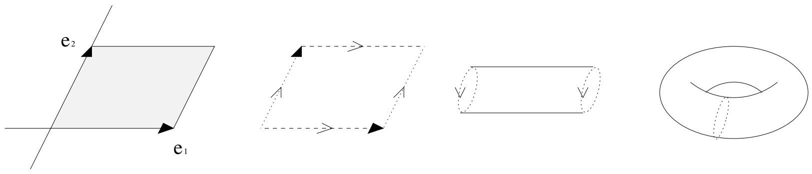


Figure 2: $\mathrm{T}^2 = \mathbb{R}^2/\Lambda$

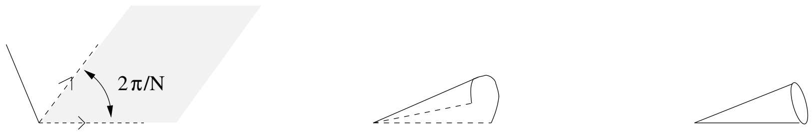


Figure 3: $\mathbb{C}/\mathbb{Z}_N$

For $g \in G$ and $x \in \mathcal{M}$, the points x and gx are equivalent in the quotient. Each point is identified with its orbit under G, hence the name orbifold. The fixed points of $\mathcal{M}$ under G are singular points of $\mathcal{O}$.

Perhaps the simplest example of an orbifold is the torus $\mathrm{T}^{\mathcal{D}}$ defined as

$$\mathrm{T}^{\mathcal{D}} = \mathbb{R}^{\mathcal{D}}/\Lambda \ , \tag{4.2}$$

where Λ is a $\mathcal{D}$-dimensional lattice. Hence, in $\mathrm{T}^{\mathcal{D}}$ the points x and $x+V$, $V \in \Lambda$, are identified. In the following we denote the basis of the torus lattice by e_a, $a = 1, \cdots, \mathcal{D}$. Fig. 2 shows the case of T^2. Since the group of translations by lattice vectors acts freely, the torus has no singular points. However, when the discrete group leaves fixed points, the orbifold has singular points. A simple example is the cone obtained by taking the quotient of $\mathbb{C} \simeq \mathbb{R}^2$ by $\mathbb{Z}_N$ generated by multiplication by $e^{2i\pi/N}$. This is shown in Fig. 3. Notice that the origin, left fixed by $\mathbb{Z}_N$, is a singular point at which there is a deficit angle $2\pi(N-1)/N$.

Since we want compact spaces we are led to consider toroidal orbifolds $\mathrm{T}^{\mathcal{D}}/G_P$, where the so called point group $G_P \subset SO(\mathcal{D})$ is a discrete group that acts crystallographically on the torus lattice Λ. The elements of G_P are rotations denoted generically θ. Alternatively, toroidal orbifolds can be expressed as $\mathbb{R}^{\mathcal{D}}/S$, where S is the so-called space group that contains rotations and translations in Λ.

The point group is the holonomy group of the toroidal orbifold [3]. To show this, take two points x and y, distinct on the torus but such that $y = \theta x + V$. Then, x and y are identified on the orbifold and moreover the tangent vectors at x are identified with the tangent vectors at y rotated by θ. Next parallel-transport some vector along a path from x to y which is closed on the orbifold. The torus is flat and hence this vector remains constant but since the tangent basis is rotated by θ, the final vector is rotated by θ with respect to the initial vector. The loop from x to y necessarily encloses a singular point since otherwise there would be no curvature to cause the non-trivial holonomy.

In the following we will mostly consider point groups $G_P = \mathbb{Z}_N$. Then $\theta^N = \mathbb{1}$ and θ has eigenvalues $e^{\pm 2i\pi v_i}$, where $v_i = k_i/N$ for some integers k_i, $i = 1, \cdots, \mathcal{D}/2$ (we take $\mathcal{D}$ even). As we mentioned before, G_P must act crystallographically on the torus lattice. This means that for $V \in \Lambda$ and $\theta \in G_P$, $\theta V \in \Lambda$. Now, since $V = n_a \mathrm{e}_a$, with integer coefficients n_a, in the lattice basis θ must be a matrix of integers. Hence, the quantities

$$\begin{aligned} \mathrm{Tr}\,\theta &= \sum_{i=1}^{\mathcal{D}/2} 2\cos 2\pi v_i \\ \chi(\theta) &= \det(1-\theta) = \prod_{i=1}^{\mathcal{D}/2} 4\sin^2 \pi v_i \end{aligned} \tag{4.3}$$

must be integers. Indeed, from Lefschetz fixed point theorem, $\chi(\theta)$ is the number of fixed points of θ. The upshot is that the requirement of crystallographic action is very restrictive. For instance, it is easy to find that for $\mathcal{D}=2$ only $N=2,3,4,6$ are allowed. In Table 1 we collect the irreducible possibilities for the v_i's when $\mathcal{D}=2,4,6$ [67]. By irreducible we mean that the corresponding θ cannot be written in a block form. Notice that the case $\mathcal{D}=2, v_1=\frac{1}{2}$ is reducible since already in a one dimensional lattice a $\mathbb{Z}_2$ (only) is allowed.

$\mathcal{D}=2$	$\mathcal{D}=4$	$\mathcal{D}=6$
(v_1)	(v_1,v_2)	(v_1,v_2,v_3)
$\frac{1}{3}(1)$	$\frac{1}{5}(1,2)$	$\frac{1}{7}(1,2,3)$
$\frac{1}{4}(1)$	$\frac{1}{8}(1,3)$	$\frac{1}{9}(1,2,4)$
$\frac{1}{6}(1)$	$\frac{1}{10}(1,3)$	$\frac{1}{14}(1,3,5)$
	$\frac{1}{12}(1,5)$	$\frac{1}{18}(1,5,7)$

Table 1: Irreducible crystallographic actions.

Given the v_i's there remains the question of finding a concrete lattice Λ that has θ^n, $n=1,\cdots,N$, as automorphisms. We refer the reader to [68, 67] for a discussion of these issues. Here we will mostly consider products of two-dimensional sub-lattices and for order two and order four rotations we take the $SO(4)$ root lattice whereas for order three and order six rotations we take the $SU(3)$ root lattice.

Let us now consider some examples.

Example 4.1: $\mathrm{T}^2(SO(4))/\mathbb{Z}_2$. Here $\mathbb{Z}_2$ has elements $\{\mathbb{1},\theta\}$, where θ is a rotation by π. As Λ we take the root lattice of $SO(4)$ with basis $\mathrm{e}_1=(1,0)$ and $\mathrm{e}_2=(0,1)$. In T^2, $\mathbb{Z}_2$ has four fixed points:

$$f_0=(0,0) \quad ; \quad f_1=(\frac{1}{2},0) \quad ; \quad f_2=(0,\frac{1}{2}) \quad ; \quad f_3=(\frac{1}{2},\frac{1}{2}) \ . \tag{4.4}$$

It is convenient to use a complex coordinate $z=x+iy$ so that $f_0=0$, $f_1=\frac{1}{2}$, $f_2=\frac{i}{2}$, $f_3=\frac{1+i}{2}$.

The steps to construct the orbifold are shown in Fig. 4. To start, we take a fundamental cell defined by vertices $(0,0),(1,0),(0,1),(1,1)$. Given the identification $x\equiv\theta^n x+V$, we observe that it is actually enough to retain half of the fundamental cell, for instance the rectangle with vertices at f_0, f_1, i and $\frac{1}{2}+i$. Furthermore, since the edges are identified as indicated in Fig. 4 we must fold by the line joining f_2 and f_3. The resulting orbifold has singular points precisely at the f_i, each with a deficit angle of π.

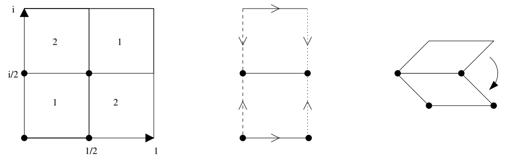


Figure 4: $\mathrm{T}^2/\mathbb{Z}_2$

Example 4.2: $\mathrm{T}^2(SU(3))/\mathbb{Z}_3$. Here $\mathbb{Z}_3$ has elements $\{\mathbb{1},\theta,\theta^2\}$, where θ is a rotation by $2\pi/3$. As Λ we take the root lattice of $SU(3)$ with basis $\mathrm{e}_1=(1,0)$ and $\mathrm{e}_2=(-\frac{1}{2},\frac{\sqrt{3}}{2})$. The fixed points of θ are

$$f_0=(0,0) \quad ; \quad f_1=(\frac{1}{2},\frac{1}{2\sqrt{3}}) \quad ; \quad f_2=(0,\frac{1}{\sqrt{3}}) \ . \tag{4.5}$$

In terms of the complex coordinate z, θ acts by multiplication by $e^{2i\pi/3}$ and the fixed points are located at 0, $\frac{1}{\sqrt{3}}e^{i\pi/6}$, $\frac{i}{\sqrt{3}}$. The element $\theta^2=\theta^{-1}$ obviously has the same fixed points. In this case the resulting orbifold has singularities at the three fixed points, each with deficit angle $4\pi/3$.

In examples 4.1 and 4.2 the total deficit angle is 4π, i.e. the orbifold is topologically an S^2, as it is also clear from Fig. 4.

Example 4.3: $\mathrm{T}^4(SO(4)^2)/\mathbb{Z}_2$. We take $\mathrm{T}^4 = \mathrm{T}^2 \times \mathrm{T}^2$ and Λ the product of two 2-dimensional square $SO(4)$ root lattices. The $\mathbb{Z}_2$ action is just a rotation by π degrees in each square sub-lattice. In terms of $z^j = x^j + iy^j$ this means

$$\mathbb{Z}_2: \quad (z^1, z^2) \to (-z^1, -z^2) \ . \tag{4.6}$$

In each sub-lattice there are four fixed points with complex coordinates $0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}$. Altogether the orbifold has then sixteen singular points.

Notice that there are no $\mathbb{Z}_2$ invariant $(1,0)$ harmonic forms and only one invariant $(2,0)$ harmonic form, namely $dz^1 \wedge dz^2$. This is an indication that the holonomy group of the orbifold is a subgroup of $SU(2)$. It turns out that the orbifold singularities at the fixed points can be 'repaired' or 'blown up' to produce a smooth manifold of $SU(2)$ holonomy, namely a smooth K3 [69]. Roughly, the idea is to excise the singular points and replace them by plugs that patch the holes smoothly. More precisely, the plugs are asymptotically Euclidean spaces (ALE) with metrics of $SU(2)$ holonomy that happen to be Eguchi-Hanson spaces. The claim that the resulting space is a smooth K3 manifold can be supported by a computation of the Hodge numbers of K3 in the orbifold picture. Firstly, the orbifold inherits the forms of T^4 that are invariant under $\mathbb{Z}_2$. Thus, the following are also harmonic forms on $\mathrm{T}^4/\mathbb{Z}_2$:

$$1 \quad , \quad dz^i \wedge d\bar{z}^j \quad , \quad dz^1 \wedge dz^2 \quad , \quad d\bar{z}^1 \wedge d\bar{z}^2 \quad , \quad dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2 \ . \tag{4.7}$$

Secondly, the blowing up process gives a contribution of sixteen to $h^{1,1}$, one from the Eguchi-Hanson Kähler form at each fixed point. Then, altogether $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$ and $h^{1,1} = 20$.

Example 4.4: $\mathrm{T}^6(SU(3)^3)/\mathbb{Z}_3$. We take $\mathrm{T}^6 = \mathrm{T}^2 \times \mathrm{T}^2 \times \mathrm{T}^2$ and Λ the product of three $SU(3)$ root lattices. The $\mathbb{Z}_3$ group is generated by an order three rotation in each sub-lattice. In terms of complex coordinates the $\mathbb{Z}_3$ action is

$$(z^1, z^2, z^3) \to (e^{2i\pi/3} z^1, e^{2i\pi/3} z^2, e^{-4i\pi/3} z^3) \ . \tag{4.8}$$

In each sub-lattice there are three fixed points located at $0, \frac{1}{\sqrt{3}} e^{i\pi/6}, \frac{i}{\sqrt{3}}$. The full orbifold has thus 27 singular points.

The singular points can be repaired to obtain a smooth manifold, the so-called Z-manifold that is a CY_3 [2]. The (3,0) harmonic form that must exist in every CY_3 is simply $dz^1 \wedge dz^2 \wedge dz^3$ that is $\mathbb{Z}_3$ invariant. The interesting Hodge numbers are computed as follows. Clearly, the nine $dz^i \wedge d\bar{z}^j$ forms are $\mathbb{Z}_3$ invariant. There are no (1,2) invariant forms on T^6. The blowing up process adds 27 (1,1) harmonic forms. Then, $h^{1,1} = 9 + 27$, $h^{1,2} = 0$ and $\chi = 72$.

To end this section we would like to address the question whether string compactification on a given orbifold can give a supersymmetric theory in the lower dimensions. We consider $\mathcal{D} = 6$, the results for $\mathcal{D} = 2, 4$ come as by-products. According to our discussion in section 2.2, supersymmetry requires the existence of covariantly constant spinors. This means that there must exist spinors ϵ such that $\theta\epsilon = \epsilon$. In our case θ is an $SO(6)$ rotation with eigenvalues $e^{\pm 2i\pi v_i}$ acting on the vector representation that has weights $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. In fact, we can write θ as

$$\theta = \exp(2\pi i(v_1 J_{12} + v_2 J_{34} + v_3 J_{56})) \ , \tag{4.9}$$

where the $J_{2i-1,2i}$ are the generators of the Cartan subalgebra. Now, since spinor weights of $SO(6)$ are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, in this representation θ has eigenvalues $e^{i\pi(\pm v_1 \pm v_2 \pm v_3)}$. Hence, to have invariant spinors we need

$$\pm v_1 \pm v_2 \pm v_3 = 0 \bmod 2 \tag{4.10}$$

for some choice of signs. This condition guarantees that the holonomy group is contained in $SU(3)$. The additional condition $N(v_1+v_2+v_3)=0 \bmod 2$, which follows from modular invariance, is derived in Appendix C. When $v_3=0$, from Table 1 we find that the only solutions are $v_1=-v_2=1/N$, $N=2,3,4,6$. The case $N=2$ is example 4.3 above, for other N's the corresponding orbifolds of T^4 are also singular limits of K3. For orbifolds of T^6, we can again use the data in Table 1 together with (4.10) to obtain all the allowed inequivalent solutions shown in Table 2 that were first found in [3]. The resulting $\mathrm{T}^6/\mathbb{Z}_N$ orbifolds are generalizations of Calabi-Yau three-folds. In all cases it can be proved that the singular points can be resolved to obtain smooth manifolds of $SU(3)$ holonomy [68, 71].

$\mathbb{Z}_3$	$\frac{1}{3}(1,1,-2)$	$\mathbb{Z}_6'$	$\frac{1}{6}(1,-3,2)$	$\mathbb{Z}_8'$	$\frac{1}{8}(1,-3,2)$
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$	$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$	$\mathbb{Z}_{12}$	$\frac{1}{12}(1,-5,4)$
$\mathbb{Z}_6$	$\frac{1}{6}(1,1,-2)$	$\mathbb{Z}_8$	$\frac{1}{8}(1,3,-4)$	$\mathbb{Z}_{12}'$	$\frac{1}{12}(1,5,-6)$

Table 2: Supersymmetric $\mathbb{Z}_N$ actions.

4.2 Orbifold Hilbert space

In this section we wish to discuss some general aspects of the propagation of closed strings on orbifolds [3]. We will explain how to determine the states belonging to the physical Hilbert space, taking into account a projection on states invariant under the orbifold group, as well as including twisted sectors.

Let $X^m(\sigma^0,\sigma^1)$, $m=1,\cdots\mathcal{D}$, be bosonic coordinates depending on the world-sheet time and space coordinates σ^0 and σ^1. Since the string is closed, σ^1 is periodic, we take its length to be 2π. We assume that $\mathcal{M}$ is flat so that before taking the quotient to obtain the orbifold, X^m satisfies the free wave equation

$$(\partial_0^2-\partial_1^2)X^m=0\ . \tag{4.11}$$

Furthermore, there are boundary conditions

$$X^m(\sigma^0,\sigma^1+2\pi)=X^m(\sigma^0,\sigma^1)\ . \tag{4.12}$$

The equations of motion follow from the action

$$S=\int d^2\sigma\,\mathcal{L}=-\frac{1}{4\pi\alpha'}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha X^m\partial_\beta X_m\ . \tag{4.13}$$

This is the Polyakov action (1.1) in flat space-time and in conformal gauge $h_{\alpha\beta}=\eta_{\alpha\beta}=\mathrm{diag}(-1,1)$. The canonical conjugate momentum is $\Pi_m=\partial\mathcal{L}/\partial(\partial_0X^m)$. In the following we will drop the index m to simplify notation. The generator of translations in X is $P=\int_0^{2\pi}d\sigma^1\Pi$.

Now, in the orbifold we know that each point is identified with its orbit under $g\in G$. Hence, as physical states we should consider only the sub-space invariant under the action of g. The appropriate projection operator is

$$\mathcal{P}=\frac{1}{|G|}\sum_{g\in G}\bar{g}\ , \tag{4.14}$$

where $\bar{g}$ is the realization of g on the string states.

Exercise 4.1: Show that $\mathcal{P}^2=\mathcal{P}$.

For example, consider the quotient of $\mathbb{R}^{\mathcal{D}}$ by translations in a lattice Λ to obtain $\mathrm{T}^{\mathcal{D}}$. Since the generator of space-time translations is the momentum P, to each $W\in\Lambda$ the operator acting on states is $e^{2\pi iP\cdot W}$ (the factor of 2π is for convenience). Then, the sub-space of invariant states contains only strings whose center of mass momentum (the eigenvalue of P) belongs to the dual lattice Λ^*. Indeed, notice that $\sum_{W\in\Lambda} e^{2\pi i P\cdot W}$ vanishes unless $P \in \Lambda^*$. Recall that Λ^* is the set of all vectors that have integer scalar product with any vector in Λ. In this case $|G|$ is equal to the volume $\mathrm{Vol}(\Lambda)$ of the unit cell of Λ. It can be shown that $\mathrm{Vol}(\Lambda)\mathrm{Vol}(\Lambda^*) = 1$.

In the orbifoldized theory there appear naturally *twisted* sectors in which X closes up to a transformation $h \in G$. This is:

$$X(\sigma^0, \sigma^1 + 2\pi) = hX(\sigma^0, \sigma^1) \ . \tag{4.15}$$

The *untwisted* sector has $h = \mathbb{1}$. In the example of $\mathrm{T}^{\mathcal{D}}$, the twisted sectors have boundary conditions

$$X(\sigma^0, \sigma^1 + 2\pi) = X(\sigma^0, \sigma^1) + 2\pi W, \quad , \quad W \in \Lambda \ . \tag{4.16}$$

Thus, the twisted sectors are just the winding sectors in which the string wraps around the torus cycles.

The twisted states must be included in order to ensure modular invariance. It is instructive to see this in the $\mathrm{T}^{\mathcal{D}}$ compactification. To begin, consider the solution to (4.11) together with (4.16). Left and right moving modes are independent so that $X = X_L + X_R$, with expansions

$$\begin{aligned} X_L(\sigma^0, \sigma^1) &= x_L + P_L(\sigma^0 + \sigma^1) + i \sum_{n\neq 0} \frac{\alpha_n}{n}\, e^{-in(\sigma^0+\sigma^1)} \\ X_R(\sigma^0, \sigma^1) &= x_R + P_R(\sigma^0 - \sigma^1) + i \sum_{n\neq 0} \frac{\tilde\alpha_n}{n}\, e^{-in(\sigma^0-\sigma^1)} \ , \end{aligned} \tag{4.17}$$

where

$$(P_L, P_R) = (P + \frac{W}{2}, P - \frac{W}{2}) \quad , \quad P \in \Lambda^* \quad , \quad W \in \Lambda \ . \tag{4.18}$$

For simplicity we are setting $\alpha' = 2$ everywhere. The Fourier coefficients α_n and $\tilde\alpha_n$ are commonly called *oscillator modes*. Quantization proceeds in the standard way by promoting the expansion coefficients to operators and imposing equal time canonical commutation relations that imply $[\alpha_m, \alpha_n] = m\delta_{m,-n}$, $[\tilde\alpha_m, \tilde\alpha_n] = m\delta_{m,-n}$. Furthermore, $[x_L, P_L] = i$ and $[x_R, P_R] = i$. It is convenient to introduce the occupation number operators

$$\mathcal{N}_L = \sum_{n=1}^{\infty} \alpha_{-n}\, \alpha_n \quad , \quad \mathcal{N}_R = \sum_{n=1}^{\infty} \tilde\alpha_{-n}\, \tilde\alpha_n \ . \tag{4.19}$$

The vacuum state $|0, 0, k_L, k_R\rangle$ is defined to be annihilated by $\alpha_n, \tilde\alpha_n,\ n > 0$, and to be an eigenvector of the momenta (P_L, P_R) with eigenvalues (k_L, k_R) of the form (4.18). Acting on the vacuum with creation operators $\alpha_{-n}, \tilde\alpha_{-n},\ n > 0$, gives states $|N_L, N_R, k_L, k_R\rangle$ that have generic eigenvalues N_L and N_R of the occupation number operators. For instance, $(\alpha_{-n_1})^{\ell_1}\, (\tilde\alpha_{-n_2})^{\ell_2}\, |0, 0, k_L, k_R\rangle$ has $N_L = n_1\ell_1$ and $N_R = n_2\ell_2$.

The Hamiltonian is

$$H = \int_0^{2\pi} d\sigma^1 (\Pi \cdot \partial_0 X - \mathcal{L}) = \frac{1}{8\pi} \int_0^{2\pi} d\sigma^1 \left[(\partial_0 X)^2 + (\partial_1 X)^2\right] \ . \tag{4.20}$$

Substituting the expansions (4.17) then gives

$$H = \frac{P_L^2}{2} + \frac{P_R^2}{2} + \mathcal{N}_L + \mathcal{N}_R - \frac{\mathcal{D}}{12} \ . \tag{4.21}$$

The constant term comes from normal ordering all annihilation operators to the right and using the analytical continuation of the zeta function to regularize the sum $\sum_{n=1}^{\infty} n = \zeta(-1) = -1/12$. The Hamiltonian is the generator of translations in σ^0, meaning that $[H, X] = -i\partial_0 X$. The generator of translations in σ^1 is

$$P_\sigma = \int_0^{2\pi} d\sigma^1 \Pi \cdot \partial_1 X = \frac{P_L^2}{2} - \frac{P_R^2}{2} + \mathcal{N}_L - \mathcal{N}_R \ . \tag{4.22}$$

Both H and P_σ can be written in terms of left and right moving Virasoro generators as

$$H = L_0 + \tilde{L}_0 \quad , \quad P_\sigma = L_0 - \tilde{L}_0 \ . \tag{4.23}$$

Then,

$$L_0 = \frac{P_L^2}{2} + \mathcal{N}_L - \frac{\mathcal{D}}{24} \quad ; \quad \tilde{L}_0 = \frac{P_R^2}{2} + \mathcal{N}_R - \frac{\mathcal{D}}{24} \ . \tag{4.24}$$

Since $\mathcal{D}$ free bosons have central charge $c = \mathcal{D}$, the constant term is the expected $-c/24$. The eigenvalue of L_0 ($\tilde{L}_0$) is the squared mass m_L^2 (m_R^2) of the given state. Invariance under translations along the closed string requires that P_σ vanishes acting on states. This implies the level-matching condition $m_R^2 = m_L^2$.

We next consider the partition function defined as

$$\mathcal{Z}(\tau, \bar{\tau}) = \mathrm{Tr}\, q^{L_0}\, \bar{q}^{\tilde{L}_0} \quad ; \quad q \equiv e^{2i\pi\tau} \quad ; \quad \tau \in \mathbb{C} \ , \tag{4.25}$$

where the trace is taken over the states $|N_L, N_R, k_L, k_R\rangle$. Knowing the spectrum we can simply compute $\mathcal{Z}(\tau, \bar{\tau})$ by counting the number of states at each level of $L_0, \tilde{L}_0$. For the toroidal compactification one finds

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^{2\mathcal{D}}} \sum_{P \in \Lambda^*} \sum_{W \in \Lambda} q^{\frac{1}{2}(P + \frac{W}{2})^2}\, \bar{q}^{\frac{1}{2}(P - \frac{W}{2})^2} \ . \tag{4.26}$$

The Dedekind eta function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k) \ , \tag{4.27}$$

arises from the contribution of the oscillator modes.

Exercise 4.2: Show (4.26).

The partition function (4.26) has the remarkable property of being invariant under the modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad ; \quad a, b, c, d \in \mathbb{Z} \quad ; \quad ad - bc = 1 \ . \tag{4.28}$$

The $SL(2, \mathbb{Z})$ modular group is generated by the transformations $\mathcal{T} : \tau \to \tau + 1$ and $\mathcal{S} : \tau \to -1/\tau$. Invariance of (4.26) under $\mathcal{T}$ follows simply because $2P \cdot W$ = even. Invariance under $\mathcal{S}$ arises only because the partition function includes a sum over windings.

Exercise 4.3: Prove invariance of (4.26) under $\mathcal{S}$ using the property $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$ and the Poisson resummation formula

$$\sum_{W \in \Lambda} e^{-\pi a (W+U)^2}\, e^{2i\pi Y \cdot (W+U)} = \frac{1}{\mathrm{Vol}(\Lambda)\, a^{\mathcal{D}/2}} \sum_{P \in \Lambda^*} e^{-\frac{\pi}{a}(P+Y)^2}\, e^{-2i\pi P \cdot U} \ , \tag{4.29}$$

where U and Y are arbitrary vectors and a is a positive constant.

Physically, the partition function $\mathcal{Z}(\tau, \bar{\tau})$ corresponds to the vacuum to vacuum string amplitude at one-loop. In this case the world-sheet surface is a torus T^2 that has precisely τ as modular parameter. From the brief discussion after (2.27) recall that T^2 with modular parameter $\tau = \tau_1 + i\tau_2$ can be defined by identifications in a lattice with basis $\mathrm{e}_1 = (1, 0)$, $\mathrm{e}_2 = (\tau_1, \tau_2)$. We can picture the T^2 as formed by a cylinder of length τ_2 in which we identify the string at the initial end with the string at the final end after translating by τ_1. Indeed, using (4.23) we find

$$\mathcal{Z}(\tau, \bar{\tau}) = \mathrm{Tr}\, e^{-2\pi\tau_2 H}\, e^{2i\pi\tau_1 P_\sigma} \ . \tag{4.30}$$

The first term in the trace is precisely what we expect of a partition function for a system propagating for Euclidean time $2\pi\tau_2$. The second term reflects a translation by $2\pi\tau_1$ in the coordinate σ^1 along the string. Now, the modular transformations (4.28) just correspond to an integral change of basis in the T^2

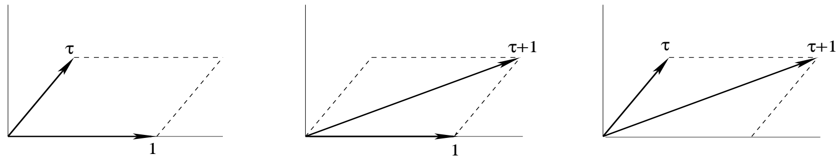


Figure 5: Three equivalent T^2 lattices

lattice. For example, Fig. 5 shows three equivalent lattices for T^2. All tori with τ's related by modular transformations are conformally equivalent and the partition function must therefore remain invariant.

In the example of toroidal compactification, the partition function is modular invariant only because the winding sectors are included. Indeed, $\mathcal{S}$ basically exchanges σ^0 and σ^1 so it transforms quantized momenta into windings. In general, the partition function for orbifold compactification is modular invariant only if twisted sectors are included. To see this, we start with the untwisted sector and implement the projection on invariant states according to (4.14). The partition function in the untwisted sector then becomes

$$\mathcal{Z}_{\mathbb{1}}(\tau,\bar\tau) = \mathrm{Tr}\,(\mathcal{P}\,q^{L_0(\mathbb{1})}\,\bar q^{\tilde L_0(\mathbb{1})}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\,(\bar g\,q^{L_0(\mathbb{1})}\,\bar q^{\tilde L_0(\mathbb{1})}) \ . \tag{4.31}$$

Due to the insertion of $\bar g$, the traces in the sum above are over states that satisfy not only the untwisted boundary condition (4.12) but also

$$X(\sigma^0+2\pi\tau_2, \sigma^1+2\pi\tau_1) = gX(\sigma^0,\sigma^1) \ . \tag{4.32}$$

We can then write schematically

$$\mathcal{Z}_{\mathbb{1}}(\tau,\bar\tau) = \frac{1}{|G|}\sum_{g\in G}\mathcal{Z}(\mathbb{1},g) \ , \tag{4.33}$$

where $\mathcal{Z}(h,g)$ means partition function with boundary conditions (4.15) in σ^1 and (4.32) in σ^0. Now, under modular transformations the boundary conditions do change. For instance, under $\mathcal{T}:\tau\to\tau+1$, $(h,g)\to(h,gh)$, and under $\mathcal{TST}:\tau\to\tau/(\tau+1)$, $(h,g)\to(gh,g)$, as implied by the change of basis depicted in Fig. 5. Then, under $\mathcal{S}:\tau\to -1/\tau$, $(h,g)\to(g,h^{-1})$ and in particular $\mathcal{S}$ transforms the untwisted sector into a twisted sector. To obtain a modular invariant partition function we must include all sectors. More precisely, for Abelian G the full partition function has the form

$$\begin{aligned}\mathcal{Z}(\tau,\bar\tau) &= \frac{1}{|G|}\sum_{h\in G}\sum_{g\in G}\mathcal{Z}(h,g)\\ &= \sum_{h\in G}\left[\frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\,(\bar g\,q^{L_0(h)}\,\bar q^{\tilde L_0(h)})\right] \ .\end{aligned} \tag{4.34}$$

The sum over h is a sum over twisted sectors while the sum over g implements the orbifold projection in each sector. For non-Abelian G we only sum over h and g such that $[h,g]=0$ since otherwise (4.15) and (4.32) are incompatible.

4.3 Bosons on $T^{\mathcal{D}}/\mathbb{Z}_N$

We now wish to derive the partition function for bosonic coordinates compactified on $T^{\mathcal{D}}/\mathbb{Z}_N$, with $\mathbb{Z}_N$ generated by θ as described in section 4.1, and torus lattice Λ. We consider *symmetric* orbifolds in which θ acts equally on left and right movers. As we have explained, we need to include sectors twisted by $\theta^k, k=0,\cdots,N-1$, in which the boundary conditions are

$$X(\sigma^0,\sigma^1+2\pi) = \theta^k X(\sigma^0,\sigma^1) + 2\pi V \quad ; \quad V\in\Lambda \ . \tag{4.35}$$

The X's still satisfy the free equations of motion (4.11) so they have mode expansions of the form

$$X(\sigma^0,\sigma^1) = X_0 + 2P\sigma^0 + W\sigma^1 + \text{oscillators} . \tag{4.36}$$

To simplify the analysis we will assume that θ^k leaves no invariant directions so that the boundary conditions generically do not allow quantized momenta nor windings in the expansion. For the center of mass coordinate X_0 we find that it must satisfy $(1-\theta^k)X_0 = 0$ modulo $2\pi\Lambda$ which just means that X_0 is a fixed point of θ^k.

To find out the effect on the oscillator modes it is useful to define complex coordinates $z^j = \frac{1}{\sqrt{2}}(X^{2j-1} + iX^{2j})$, $j = 1,\cdots,\mathcal{D}/2$, such that $\theta z^j = e^{2i\pi v_j} z^j$ as we have seen in section 4.1. Next write the z^j expansion as

$$z^j(\sigma^0,\sigma^1) = z_0^j + i\sum_t \frac{\alpha_t^j}{t}\, e^{-it(\sigma^0+\sigma^1)} + i\sum_s \frac{\tilde{\alpha}_s^j}{s}\, e^{-is(\sigma^0-\sigma^1)} , \tag{4.37}$$

where the frequencies t and s are to be determined by imposing the boundary condition (4.35). In this way we obtain $e^{-2i\pi t} = e^{2i\pi k v_j}$ and then $t = n - kv_j$, with n integer. Likewise, $s = n + kv_j$. For the complex conjugate $\bar{z}^j$ there is an analogous expansion with coefficients $\bar{\alpha}^j_{n+kv_j}$ and $\bar{\tilde{\alpha}}^j_{n-kv_j}$. Let us focus on the left-movers. After quantization, $[\bar{\alpha}^i_{m+kv_i}, \alpha^j_{n-kv_j}] = (m+kv_j)\delta^{i,j}\delta_{m,-n}$, with other commutators vanishing. There are now several Fock vacua $|f,0\rangle_k$, where $f = 1,\cdots,\chi(\theta^k)$, is the fixed point label. Each vacuum is annihilated by all positive-frequency modes. The creation operators are thus $\alpha^j_{-kv_j}$, $\alpha^j_{-1-kv_j},\cdots$ and $\bar{\alpha}^j_{-1+kv_j}$, $\bar{\alpha}^j_{-2+kv_j},\cdots$ (assuming $0 < kv_j < 1$). The occupation number operator is

$$\mathcal{N}_L = \sum_{n=-\infty}^{\infty} :\alpha^j_{-n-kv_j}\,\bar{\alpha}^j_{n+kv_j}: , \tag{4.38}$$

where :: means normal ordering, i.e. all positive-frequency modes to the right. For right-movers the results are analogous.

We now construct the partition function that according to (4.34) has the form

$$\begin{aligned} \mathcal{Z} &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \mathcal{Z}(\theta^k,\theta^\ell) , \\ \mathcal{Z}(\theta^k,\theta^\ell) &= \mathrm{Tr}\,(\theta^\ell\, q^{L_0(\theta^k)}\,\bar{q}^{\tilde{L}_0(\theta^k)}) . \end{aligned} \tag{4.39}$$

The strategy is to start with the untwisted sector ($k=0$) in which the Virasoro operators $L_0(\mathbb{1})$ and $\tilde{L}_0(\mathbb{1})$ are those given in (4.24). In particular, $\mathcal{Z}(\mathbb{1},\mathbb{1})$ is just (4.26). For $\ell \neq 0$ we need to evaluate the trace with the θ^ℓ insertion. Since we are assuming that θ^ℓ leaves no unrotated directions, neither quantized momenta nor windings survive the trace. We only need to consider states obtained from the Fock vacuum by acting with creation operators which for the complex coordinates are eigenvectors of θ^ℓ. The Fock vacuum, denoted $|0\rangle_0$, is defined to be invariant under θ. Then, for instance, for the left movers in z^j we find the contribution

$$\mathrm{Tr}\,(\theta^\ell\, q^{L_0^j(\mathbb{1})}) = q^{-1/12}(1 + qe^{2i\pi\ell v_j} + qe^{-2i\pi\ell v_j} + \cdots) . \tag{4.40}$$

The first term comes from $|0\rangle_0$, the next two from states with α^j_{-1} and $\bar{\alpha}^j_{-1}$ acting on $|0\rangle_0$, and so on. In fact, the whole expansion can be cast as

$$\mathrm{Tr}\,(\theta^\ell\, q^{L_0^j(\mathbb{1})}) = q^{-1/12}\prod_{n=1}^{\infty}(1-q^n e^{2i\pi\ell v_j})^{-1}\,(1-q^n e^{-2i\pi\ell v_j})^{-1} . \tag{4.41}$$

This result can be conveniently written by using Jacobi ϑ functions that have the product representation

$$\frac{\vartheta[^{\delta}_{\varphi}](\tau)}{\eta(\tau)} = e^{2i\pi\delta\varphi}\, q^{\frac{1}{2}\delta^2-\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^{n+\delta-\frac{1}{2}}e^{2i\pi\varphi})\,(1+q^{n-\delta-\frac{1}{2}}e^{-2i\pi\varphi}) . \tag{4.42}$$

Then,

$$\mathrm{Tr}\,(\theta^\ell\, q^{L_0^j(\mathbb{1})}) = -2\sin\ell\pi v_j \frac{\eta(\tau)}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+\ell v_j}](\tau)}\ . \tag{4.43}$$

Notice that for $\ell = 0$, (4.41) becomes $1/\eta^2$, as it should. Taking into account left and right movers for all coordinates we obtain

$$\mathcal{Z}(\mathbb{1},\theta^\ell) = \chi(\theta^\ell)\left|\prod_{j=1}^{\mathcal{D}/2}\frac{\eta}{\vartheta[{\frac{1}{2} \atop \frac{1}{2}+\ell v_j}]}\right|^2\ , \tag{4.44}$$

where $\chi(\theta^\ell) = \prod_{j=1}^{\mathcal{D}/2} 4\sin^2\pi\ell v_j$ is the number of fixed points of θ^ℓ, cf. (4.3). We remark, as it is clear from (4.41), that the coefficient of the first term in the expansion in (4.44) is actually one. This means that in the full untwisted sector, i.e. fixing $k = 0$ and summing over ℓ, the untwisted vacuum appears with the correct multiplicity one.

To obtain other pieces $\mathcal{Z}(\theta^k, \theta^\ell)$ we take advantage of modular invariance. For example, $\mathcal{Z}(\theta^k, \mathbb{1})$ simply follows applying $\tau \to -1/\tau$ to (4.44). Using the modular properties of ϑ functions given in (C.4) gives

$$\begin{aligned}\mathcal{Z}(\theta^k,\mathbb{1}) &= \chi(\theta^k)\left|\prod_{j=1}^{\mathcal{D}/2}\frac{\eta}{\vartheta[{\frac{1}{2}+kv_j \atop \frac{1}{2}}]}\right|^2 \\ &= \chi(\theta^k)\,(q\bar{q})^{-\frac{\mathcal{D}}{24}+E_k}\left|\prod_{j=1}^{\mathcal{D}/2}\prod_{n=1}^{\infty}(1-q^{n-1+kv_j})^{-1}\,(1-q^{n-kv_j})^{-1}\right|^2\ ,\end{aligned} \tag{4.45}$$

where E_k is the twisted oscillator contribution to the zero point energy given by

$$E_k = \sum_{j=1}^{\mathcal{D}/2}\frac{1}{2}kv_j(1-kv_j)\ . \tag{4.46}$$

When $kv_j > 1$ we must substitute $kv_j \to (kv_j - 1)$ in (4.46).

Exercise 4.4: Derive (4.45).

The lowest order term in the expansion (4.45) does have coefficient $\chi(\theta^k)$ in agreement with the fact that in the θ^k sector the center of mass coordinate can be any fixed point. The q expansion also shows the contribution of the states created by operators $\alpha^j_{-kv_j}, \alpha^j_{-1-kv_j}, \cdots$ and $\bar{\alpha}^j_{-1+kv_j}, \bar{\alpha}^j_{-2+kv_j}, \cdots$. In fact, from the exponents of q we can read off the eigenvalues of $L_0(\theta^k)$, i.e. the squared masses $m_L^2(\theta^k)$. The general result can be written as

$$m_L^2(\theta^k) = N_L + E_k - \frac{\mathcal{D}}{24}\ . \tag{4.47}$$

Here N_L is the occupation number of the left-moving oscillators. For example, $\alpha^j_{-kv_j}|0\rangle_k$ and $\bar{\alpha}^j_{-1+kv_j}|0\rangle_k$ have $N_L = kv_j$ and $N_L = 1 - kv_j$, respectively. In the untwisted sector, or more generically in sectors in which quantized momenta or windings are allowed, m_L^2 also includes a term of the form $\frac{1}{2}P_L^2$. For particular shapes of the torus, $\frac{1}{2}P_L^2$ can precisely lead to extra massless states that signal enhanced symmetries as in the well known example of circle compactification at the self-dual radius. In these notes we will assume a generic point in the torus moduli space so that $\frac{1}{2}P_L^2$ does not produce new massless states. For right movers, $m_R^2(\theta^k)$ is completely analogous to (4.47). Notice that the level-matching condition becomes $N_L = N_R$.

We can continue generating pieces of the partition function by employing modular transformations. For example, applying $\tau \to \tau + 1$ to (4.45) gives $\mathcal{Z}(\theta^k, \theta^k)$. The general result can be written as

$$\mathcal{Z}(\theta^k,\theta^\ell) = \chi(\theta^k,\theta^\ell)\left|\prod_{j=1}^{\mathcal{D}/2}\frac{\eta}{\vartheta[{\frac{1}{2}+kv_j \atop \frac{1}{2}+\ell v_j}]}\right|^2\ , \tag{4.48}$$

where $\chi(\theta^k,\theta^\ell)$ is the number of simultaneous fixed points of θ^k and θ^ℓ. This formula is valid when θ^k leaves no fixed directions, otherwise a sum over momenta and windings could appear. This is important when determining the $\mathbb{Z}_N$-invariant states [67]. The correct result can be found by carefully determining the untwisted sector pieces and then performing modular transformations.

Exercise 4.5: Use (C.4) to show that (4.48) has the correct modular transformations, i.e. $\mathcal{Z}(\theta^k,\theta^\ell)$ transforms into $\mathcal{Z}(\theta^k,\theta^{k+\ell})$ under $\mathcal{T}$ and into $\mathcal{Z}(\theta^\ell,\theta^{-k})$ under $\mathcal{S}$.

Let us now describe the spectrum in a θ^k twisted sector. States are chains of left and right moving creation operators acting on the vacuum. Schematically this is

$$\alpha\cdots\bar{\alpha}\cdots\tilde{\alpha}\cdots\bar{\tilde{\alpha}}\cdots|f,0\rangle_k \ . \tag{4.49}$$

Level-matching $N_L = N_R$ must be satisfied. States are further characterized by their transformation under a $\mathbb{Z}_N$ element, say θ^ℓ. The oscillator piece is just multiplied by an overall phase $e^{2i\pi\ell\rho}$, where $\rho=\rho_L+\rho_R$. In turn ρ_L (ρ_R) is found by adding the phases of all left (right) modes in (4.49). Concretely, each left-moving oscillator $\alpha^j_{-kv_j}$, $\alpha^j_{-1-kv_j},\cdots$ (coming from z^j) adds v_j to ρ_L, whereas each $\bar{\alpha}^j_{-1+kv_j}$, $\bar{\alpha}^j_{-2+kv_j},\cdots$ (coming from $\bar{z}^j$) contributes $-v_j$ to ρ_L. For right-movers, each mode $\bar{\tilde{\alpha}}^j_{-kv_j}$, $\bar{\tilde{\alpha}}^j_{-1-kv_j},\cdots$, contributes $-v_j$ to ρ_R and each $\tilde{\alpha}^j_{-1+kv_j}$, $\tilde{\alpha}^j_{-2+kv_j},\cdots$ adds v_j to ρ_R. Finally, the action on the fixed points must be $\theta^\ell|f,0\rangle_k=|f',0\rangle_k$, where f' is also a fixed point of θ^k.

Only states invariant under the full $\mathbb{Z}_N$ action survive in the spectrum. For example, in the untwisted sector ($k=0$), both $\alpha^1_{-1}\tilde{\alpha}^1_{-1}|0\rangle_0$ and $\bar{\alpha}^1_{-1}\tilde{\alpha}^1_{-1}|0\rangle_0$ have $N_L=N_R=1$ but the first is not invariant because it picks up a phase $e^{4i\pi v_1}$ under θ. For $k\neq 0$ there is a richer structure because states sit at fixed points. In the θ sector, $\chi(\theta,\theta^\ell)=\chi(\theta)$, i.e. all θ^ℓ leave the fixed points of θ invariant. Hence, $|f,0\rangle_1$ and chain states (4.49) with $\rho_L+\rho_R=0$ are invariant $\forall f$, meaning that there is one such state at each fixed point of θ. For N odd, $\chi(\theta^k,\theta^\ell)=\chi(\theta^k)=\chi(\theta)$, so that all twisted sectors are like the θ sector.

For N even, in general $\chi(\theta^k,\theta^\ell)$, $k\neq 1, N-1$, depends on ℓ. For example, take a $\mathrm{T}^2/\mathbb{Z}_4$ with square $SO(4)$ lattice (cf. Example 4.1) and θ a $\pi/2$ rotation ($v_1=1/4$). Then, θ^2 has the four fixed points in (4.4): f_0 and f_3 that are also fixed by θ, plus f_1 and f_2 that are exchanged by θ. Thus, in the θ^2 sector, there are three invariant vacua, namely $|f_0,0\rangle_2$, $|f_3,0\rangle_2$ and $[|f_1,0\rangle_2+|f_2,0\rangle_2]$. Likewise, any level-matched chain, e.g. $\bar{\alpha}_{-\frac12}\tilde{\alpha}_{-\frac12}$, with $\rho_L+\rho_R=0$, acting on the three vacua gives states that also survive in the spectrum. There are also invariant states of the form $\alpha_{-\frac12}\tilde{\alpha}_{-\frac12}[|f_1,0\rangle_2-|f_2,0\rangle_2]$.

Conventionally, we drop the fixed point dependence and speak of states labeled by $(N_L,\rho_L;N_R,\rho_R)$, with $N_L=N_R$ determining the mass level, and having a degeneracy factor $\mathcal{F}_k(N_L,\rho_L;N_R,\rho_R)$ that might be zero when the state is not invariant. In the $\mathrm{T}^2/\mathbb{Z}_4$ example above, there are e.g. states $|0\rangle_2$ and $\bar{\alpha}_{-\frac12}\tilde{\alpha}_{-\frac12}|0\rangle_2$ with $\mathcal{F}_2=3$, $\alpha_{-\frac12}\tilde{\alpha}_{-\frac12}|0\rangle_2$ with $\mathcal{F}_2=1$, and so on. A systematic way to determine the degeneracy factor is to implement the orbifold projection by performing the sum $\frac{1}{N}\sum_{\ell=1}^{N-1}\mathcal{Z}(\theta^k,\theta^\ell)$. Using (4.48) and (4.42) we obtain

$$\mathcal{F}_k(N_L,\rho_L;N_R,\rho_R)=\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{\chi}(\theta^k,\theta^\ell)\,e^{2i\pi\ell(\rho_L+\rho_R)} \ . \tag{4.50}$$

Here $\tilde{\chi}(\theta^k,\theta^\ell)$ is a numerical factor that counts the fixed point multiplicity. More concretely, $\tilde{\chi}(1,\theta^\ell)=1$, so that in the untwisted sector $\mathcal{F}_0$ projects out precisely the states non-invariant under θ that have $\rho_L+\rho_R$ not integer. In twisted sectors $\tilde{\chi}(\theta^k,\theta^\ell)$ is the number of simultaneous fixed points of θ^k and θ^ℓ in the sub-lattice effectively rotated by θ^k. $\tilde{\chi}(\theta^k,\theta^\ell)$ differs from $\chi(\theta^k,\theta^l)$ because when $kv_j=$ integer, the expansion of $\vartheta[{}^{\frac12+kv_j}_{\frac12+\ell v_j}]/\eta$ has a prefactor $(-2\sin\pi\ell v_j)$, as follows using (4.42). Thus, the actual coefficient in the expansion of (4.48) is $\tilde{\chi}(\theta^k,\theta^l)=\chi(\theta^k,\theta^l)/\prod_{j,kv_j\in\mathbb{Z}}4\sin^2\pi\ell v_j$.

4.4 Type II strings on toroidal $\mathbb{Z}_N$ symmetric orbifolds

The new ingredient is the presence of world-sheet fermions with boundary conditions

$$\begin{aligned} \Psi(\sigma^0, \sigma^1 + 2\pi) &= -e^{2\pi i\alpha}\theta^k \Psi(\sigma^0, \sigma^1) \ , \\ \Psi(\sigma^0 + 2\pi\tau_2, \sigma^1 + 2\pi\tau_1) &= -e^{2\pi i\beta}\theta^\ell \Psi(\sigma^0, \sigma^1) \ , \end{aligned} \tag{4.51}$$

where $\alpha, \beta = 0, \frac{1}{2}$ are the spin structures. The full partition function has the form (4.39). Each contribution to the sum is explicitly evaluated as

$$\mathcal{Z}(\theta^k, \theta^\ell) = \mathrm{Tr}_{\,(\mathrm{NS}\oplus\mathrm{R})(\mathrm{NS}\oplus\mathrm{R})}\{\mathrm{P}_{\mathrm{GSO}}\, \theta^\ell\, q^{L_0(\theta^k)}\, \bar{q}^{\tilde{L}_0(\theta^k)}\} \ . \tag{4.52}$$

The trace is over left and right Neveu-Schwarz (NS) and Ramond (R) sectors for the fermions. This is equivalent to summing over $\alpha = 0, \frac{1}{2}$. Similarly, the GSO (Gliozzi-Scherk-Olive) projection is equivalent to summing over $\beta = 0, \frac{1}{2}$ [4, 5, 6].

To find $\mathcal{Z}(\theta^k, \theta^\ell)$ we again start from the untwisted sector in which the Virasoro operators are known and then use modular invariance. The explicit form of $\mathcal{Z}(\theta^k, \theta^\ell)$ can be found in [70] and will be presented in Appendix C. It follows that the eigenvalues of $L_0(\theta^k)$ are

$$m_L^2(\theta^k) = N_L + \frac{1}{2}(r + k\, v)^2 + E_k - \frac{1}{2} \ . \tag{4.53}$$

Most terms in this formula arise as in the purely bosonic case of last section. In particular, E_k is given in (4.46). Notice that N_L and N_R also receive (integer) contributions from the fermionic degrees of freedom. The vector r is an $SO(8)$ weight as explained in Appendix C. The vector v is $(0, v_1, v_2, v_3)$, with the v_i specifying the $\mathbb{Z}_N$ action. When r belongs to the scalar or vector class, r takes the form (n_0, n_1, n_2, n_3), with n_a integer. This is the Neveu-Schwarz sector in which left-movers are space-time bosons. When r belongs to a spinorial class it takes the form $(n_0 + \frac{1}{2}, n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2})$. This is the Ramond sector in which left-movers are space-time fermions. For example, the weights of the fundamental vector and spinor representations are:

$$\begin{aligned} \mathbf{8}_v &= (\underline{\pm 1, 0, 0, 0}) \ ; \ \mathbf{8}_s = \pm(\underline{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}}) \ , \\ \mathbf{8}_c &= \{(\underline{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}), \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\} \ , \end{aligned} \tag{4.54}$$

where underlining means permutations. As explained in Appendix C, the GSO projection turns out to be $\sum r_a = \text{odd}$. Thus, in the untwisted sector, massless states must have $r^2 = 1$ and the possible solutions are $\mathbf{8}_v$ and $\mathbf{8}_s$.

For type II strings the mass formula for right-movers is completely analogous to (4.53):

$$m_R^2(\theta^k) = N_R + \frac{1}{2}(p + k\, v)^2 + E_k - \frac{1}{2} \ , \tag{4.55}$$

where p is an $SO(8)$ weight as well. In type IIB the GSO projection is also $\sum p_a = \text{odd}$ in both NS and R sectors. In type IIA one has instead $\sum p_a = \text{even}$ in the R sector. In the untwisted sector the spinor weights are then those of $\mathbf{8}_c$. Notice that upon combining left and right movers, states in (NS,NS) and (R,R) are space-time bosons, whereas states in (NS,R) and (R,NR) are space-time fermions.

States in a θ^k-twisted sector are characterized by $(N_L, \rho_L, r; N_R, \rho_R, p)$ such that the level-matching condition $m_L^2 = m_R^2$ is satisfied. Here ρ_L and ρ_R are due only to the internal bosonic oscillators as we explained in the previous section. The degeneracy factor of these states follows from the orbifold projection. Using the results in section 4.3 and Appendix C we find

$$\mathcal{F}(N_L, \rho_L, r; N_R, \rho_R, p) = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{\chi}(\theta^k, \theta^\ell)\, \Delta(k, \ell) \ , \tag{4.56}$$

where the phase Δ is

$$\Delta(k,\ell) = \exp\{2\pi\, i[(r+kv)\cdot \ell v - (p+kv)\cdot \ell v + \ell(\rho_L+\rho_R)]\} \ . \tag{4.57}$$

The factor $\tilde{\chi}(\theta^k,\theta^\ell)$ that takes into account the fixed point multiplicity was already introduced in (4.50).

Below we will consider examples of compactifications to six and four dimensions. We will find that, as expected, one obtains results similar to those found in K3 and CY_3 compactifications.

4.4.1 Six dimensions

We first consider type IIA on $\mathrm{T}^4/\mathbb{Z}_3$. As torus lattice we take the product of two $SU(3)$ root lattices. The $\mathbb{Z}_3$ action has $v = (0,0,\frac{1}{3},-\frac{1}{3})$. The resulting theory in six dimensions has (1,1) supersymmetry that has gravity and vector multiplets with structure

$$\begin{aligned} \mathcal{G}_{11}(6) &= \{g_{\mu\nu}, \psi_\mu^{(+)}, \psi_\mu^{(-)}, \psi^{(+)}, \psi^{(-)}, B_{\mu\nu}, V_\mu^a, \phi\} \quad ; \quad a = 1,\cdots 4 \ , \\ \mathcal{V}_{11}(6) &= \{A_\mu, \lambda^{(+)}, \lambda^{(-)}, \varphi^a\} \ , \end{aligned} \tag{4.58}$$

where $\lambda^{(\pm)}$ are Weyl spinors and the φ^a real scalars. Below we will see how the orbifold massless states fit into (1,1) supermultiplets.

In the untwisted sector, candidate massless states allowed by the orbifold projection (4.56) must have $r\cdot v = p\cdot v = 0, \pm 1/3$. With $r\cdot v = p\cdot v = 0$ there are

$$\begin{array}{cc} r & p \\ (\underline{\pm 1, 0}, 0, 0) & (\underline{\pm 1, 0}, 0, 0) \\ \pm(\underline{-\frac{1}{2}, \frac{1}{2}}, \frac{1}{2}, \frac{1}{2}) & \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ & \pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \end{array} \ . \tag{4.59}$$

The first two entries in r and p, corresponding to the non-compact coordinates, indicate the Lorentz representation under the little group $SO(4) \simeq SU(2)\times SU(2)$. The vector $(\underline{\pm 1, 0})$ of $SO(4)$ is the $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\times SU(2)$, whereas the spinors $(\underline{\frac{1}{2},-\frac{1}{2}})$ and $\pm(\frac{1}{2},\frac{1}{2})$ are the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations respectively. In (4.59) we thus have the product

$$\big[(\frac{1}{2},\frac{1}{2}) \oplus 2(\frac{1}{2},0)\big]_{\mathrm{left}} \otimes \big[(\frac{1}{2},\frac{1}{2}) \oplus 2(0,\frac{1}{2})\big]_{\mathrm{right}} \ . \tag{4.60}$$

It is simple to check that the product gives rise to the representations that make up the gravity supermultiplet $\mathcal{G}_{11}(6)$ in (4.58).

In the untwisted sector with $r\cdot v = p\cdot v = 1/3$ we find

$$\begin{array}{cc} r & p \\ (0,0,1,0) & (0,0,1,0) \\ (0,0,0,-1) & (0,0,0,-1) \\ (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}) & (-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}) \\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}) & (\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}) \end{array} \ . \tag{4.61}$$

In terms of little group representations we have the product

$$\big[2(0,0) \oplus (0,\frac{1}{2})\big]_{\mathrm{left}} \otimes \big[2(0,0) \oplus (\frac{1}{2},0)\big]_{\mathrm{right}} \ . \tag{4.62}$$

In this way we obtain the representations that fill a vector multiplet $\mathcal{V}_{11}(6)$. For $r\cdot v = p\cdot v = -1/3$ one also obtains a vector multiplet. In both cases the group is $U(1)$.

Let us now turn to the θ-twisted sector. Plugging v and $E_1 = 2/9$ we find that $m_R^2 = m_L^2 = 0$ implies the same r, p given in (4.61). Taking into account the fixed point multiplicity gives then 9 vector multiplets. In the θ^2 sector we find the same result.

In conclusion, type IIA compactification on $\mathrm{T}^4/\mathbb{Z}_3$ yields a (1,1) supersymmetric theory in six dimensions with one gravity multiplet and twenty vector multiplets. Other $\mathrm{T}^4/\mathbb{Z}_N$ orbifolds give exactly the same result which is also obtained in type IIA compactification on a smooth K3 manifold.

Compactification of type IIB on $\mathrm{T}^4/\mathbb{Z}_N$ follows in a similar way. We can obtain the results from the type IIA case noting that the different GSO projection for left-moving spinors simply amounts to changing the little group representation. For example, in the untwisted sector instead of (4.60) we have

$$\big[(\frac{1}{2},\frac{1}{2})\oplus 2(\frac{1}{2},0)\big]_{\mathrm{left}} \otimes \big[(\frac{1}{2},\frac{1}{2})\oplus 2(\frac{1}{2},0)\big]_{\mathrm{right}} \ . \tag{4.63}$$

In the product there are now two gravitini of the same chirality so that the resulting theory in six dimensions has $(2,0)$ supersymmetry with gravity and tensor multiplets having the field content

$$\begin{aligned} \mathcal{G}_{20}(6) &= \{g_{\mu\nu}, \psi_\mu^{a(+)}, B_{\mu\nu}^{I(+)}\} \quad ; \quad a=1,2 \quad ; \quad I=1,\cdots 5 \ , \\ \mathcal{T}_{20}(6) &= \{B_{\mu\nu}^{(-)}, \psi^{a(-)}, \varphi^I\} \ , \end{aligned} \tag{4.64}$$

where the superscript $(+)$ or $(-)$ on the antisymmetric tensors indicates whether they have self-dual or anti-self-dual field strength. Altogether the product (4.63) gives a gravity multiplet $\mathcal{G}_{20}(6)$ together with a tensor multiplet $\mathcal{T}_{20}(6)$. Other states from the untwisted sector and the twisted sectors give rise to 20 tensor multiplets. In conclusion, compactification of type IIB on $\mathrm{T}^4/\mathbb{Z}_N$ gives $(2,0)$ supergravity with 21 tensor multiplets, exactly what is found in the compactification on K3 [72].

4.4.2 Four dimensions

The resulting theory has $\mathcal{N}=2$ supersymmetry. The massless fields must belong to the gravity multiplet or to hypermultiplets and vector multiplets. Schematically, the content of these multiplets is

$$\begin{aligned} \mathcal{G}_2(4) &= \{g_{\mu\nu}, \psi_\mu^a, V_\mu\} \quad ; \quad a,b=1,2 \ , \\ \mathcal{H}_2(4) &= \{\psi^a, \varphi^{ab}\} \ , \\ \mathcal{V}_2(4) &= \{A_\mu, \lambda^a, \varphi^a\} \ . \end{aligned} \tag{4.65}$$

Note that $\mathcal{G}_2(4)$ contains the so-called graviphoton V_μ. Below we will group the orbifold massless states into these supermultiplets. We study type IIB on $\mathrm{T}^6/\mathbb{Z}_3$. The torus lattice is the product of three $SU(3)$ root lattices. The $\mathbb{Z}_3$ action has $v=(0,\frac{1}{3},\frac{1}{3},-\frac{2}{3})$.

Now the massless states are classified by the little group $SO(2)$, i.e. by helicity λ. For a given state, $\lambda=\lambda_r-\lambda_p$ where λ_r can be read from the first component of the $SO(8)$ weight r, and likewise for λ_p. In the untwisted sector, candidate massless states allowed by the orbifold projection must have $r\cdot v=p\cdot v=0,\pm 1/3$. With $r\cdot v=p\cdot v=0$ we find

$$\begin{array}{cc} r & p \\ (\pm1,0,0,0) & (\pm1,0,0,0) \\ \pm(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) & \pm(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) \end{array} \ . \tag{4.66}$$

Considering all possible combinations in (4.66) we find the helicities

$$\{\pm2, 2\times(\pm\frac{3}{2}), \pm1\} \oplus \{2\times(\pm\frac{1}{2}), 4\times(0)\} \ . \tag{4.67}$$

Comparing with the structure of the $\mathcal{N}=2$ supersymmetric multiplets in four dimensions, cf. (4.65), we observe that (4.67) includes a gravity multiplet $\mathcal{G}_2(4)$ plus a hypermultiplet $\mathcal{H}_2(4)$. The four real scalars in the hypermultiplet are the dilaton, the axion dual to $B_{\mu\nu}$, both arising from (NS,NS) (both r,p vectorial), plus a 0-form and another axion dual to $\tilde{B}_{\mu\nu}$, both arising from (R,R) (both r,p spinorial).

In the untwisted sector with $r \cdot v = p \cdot v = \pm 1/3$ we have

$$\begin{array}{ccc} & r & p \\ r \cdot v = \frac{1}{3} & (0, \underline{1, 0, 0}) & (0, \underline{1, 0, 0}) \\ & (-\frac{1}{2}, \underline{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}) & (-\frac{1}{2}, \underline{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}) \\ r \cdot v = -\frac{1}{3} & (0, \underline{-1, 0, 0}) & (0, \underline{-1, 0, 0}) \\ & (\frac{1}{2}, \underline{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}}) & (\frac{1}{2}, \underline{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}}) \end{array} \quad . \tag{4.68}$$

Evaluating the helicities of all allowed combinations we find precisely nine hypermultiplets.

Consider now the θ-twisted sector. Plugging v and $E_1 = 1/3$ we find that $m_R^2 = m_L^2 = 0$ has solutions $r, p = (0,0,0,1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. In the θ^{-1} sector the solutions are $r, p = (0,0,0,-1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. According to the orbifold projection we can then combine the following

$$\begin{array}{ccc} & r & p \\ \theta & (0,0,0,1) & (0,,0,0,1) \\ & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) & (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \\ \theta^{-1} & (0,0,0,-1) & (0,0,0,-1) \\ & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \end{array} \quad . \tag{4.69}$$

Altogether we find the degrees of freedom of one hypermultiplet. Taking into account the fixed point multiplicity shows that 27 hypermultiplets originate in the twisted sectors.

In conclusion, compactification of type IIB on $\mathrm{T}^6/\mathbb{Z}_3$ has massless content summarized by

$$\mathcal{G}_2(4) + \mathcal{H}_2(4) + 36\mathcal{H}_2(4) \ . \tag{4.70}$$

This result agrees with the general result for type IIB compactification on a CY_3 manifold. In fact, as we explained in section 4.1, the $\mathrm{T}^6/\mathbb{Z}_3$ orbifold has $h^{1,1} = 36$ and $h^{1,2} = 0$.

Compactification of type IIA on $\mathrm{T}^6/\mathbb{Z}_3$ is completely analogous. The results are easily obtained changing the left-moving spinor helicities appropriately. In the untwisted sector with $r \cdot v = p \cdot v = 0$ there are no changes. In the untwisted sector with $r \cdot v = p \cdot v = \pm 1/3$, as well as in the twisted sectors, instead of hypermultiplets there appear vector multiplets. Hence, type IIA on $\mathrm{T}^6/\mathbb{Z}_3$ has massless multiplets

$$\mathcal{G}_2(4) + \mathcal{H}_2(4) + 36\mathcal{V}_2(4) \ . \tag{4.71}$$

This agrees with the result for compactification on a CY_3.

5 Recent developments

We have discussed basic aspects of supersymmetry preserving string compactifications. The main simplifying assumption was that the only background field allowed to have a non-trivial vacuum expectation value (vev) was the metric, for which the Ansatz (2.9) was made, and a constant dilaton ϕ_0 that fixes the string coupling constant as $g_s = e^{\phi_0}$. When we considered the complexification of the Kähler cone we also allowed a vev for the (NS,NS) antisymmetric tensor B_{MN}, but limited to vanishing field strength so that the equations of motion do not change. Restricting to these backgrounds means exploring only a small subspace of the moduli space of supersymmetric string compactifications. Type II string theory has several other massless bosonic excitations, the dilaton ϕ and the (R,R) p-form fields $A^{(p)}$ with p even for type IIB and p odd for type IIA, which could get non-vanishing vevs. The interesting situation is when the vevs for the field strengths $H = dB$ and $F^{(p+1)} = dA^{(p)}$ lead to non-vanishing fluxes through non-trivial homology cycles in the internal manifold. It is clearly important to examine the implications of these fluxes. One interesting result to date is that fluxes can generate a potential for moduli scalars [73]. This provides a mechanism for lifting flat directions in moduli space.

If the additional background fields are non-trivial they will have in general a non-zero energy-momentum tensor T_{MN} that will back-react on the geometry and distort it away from the Ricci-flat Calabi-Yau metric. At the level of the low-energy effective action this means that the lowest order (in α') equation of motion for the metric is no longer the vacuum Einstein equation $R_{MN} = 0$ but rather $R_{MN} = T_{MN}$. We also have to satisfy the equations of motion of the other background fields (setting them to zero is one solution, but we are interested in less trivial ones) and the Bianchi identities of their field strengths. Again, a practical way to proceed is to require unbroken supersymmetry, i.e. to impose that the fermionic fields have vanishing supersymmetric transformations which are now modified by the presence of additional background fields, cf. (2.10). It must then be checked that the Bianchi identities and the equations of motion are satisfied.

The effect of H flux was studied early on [74] and has lately attracted renewed attention. The upshot is that the supersymmetry preserving backgrounds are, in general, not Calabi-Yau manifolds. The analysis of these solutions is a current research subject. For recent papers that give references to the previous literature see e.g. [75].

The presence of (R,R) fluxes leads to an even richer zoo of possible type II string compactifications. One simple and well-studied example is the $AdS_5 \times S^5$ solution of type IIB supergravity which has, in addition to the metric, a non-trivial five-form field strength $F^{(5)}$ background. A general analysis of type IIB compactifications to four dimensions, including backgrounds for all bosonic fields as well as D-brane and orientifold plane sources, was given in [76]. Conditions for $\mathcal{N}=1$ supersymmetry of such configurations were found in [77]. These results have been applied in recent attempts to construct realistic models with moduli stabilization [78].

Compactification of M-theory, or its low-energy effective field theory, eleven-dimensional supergravity, on manifolds of G_2 holonomy, have also been much explored lately. These compactifications lead to $\mathcal{N}=1$ supersymmetry in four dimensions and are interesting in their own right and also in relation with various string dualities, such as compactification of M-theory on a manifold with G_2 holonomy and of the heterotic string on a Calabi-Yau manifold. See [79] for a recent review.

There are many other aspects which one could mention in the context of string compactifications. It is a vast and still growing subject with many applications in physics and mathematics. We hope that our lecture notes will be of use for those who are just entering this interesting and fascinating field.

Appendix A: Conventions and definitions

A.1: Spinors

The Dirac matrices Γ^A, $A = 0, \cdots, D-1$, satisfy the Clifford algebra

$$\{\Gamma^A, \Gamma^B\} \equiv \Gamma^A\Gamma^B + \Gamma^B\Gamma^A = 2\eta^{AB} , \tag{A.1}$$

where $\eta^{AB} = \mathrm{diag}(-1, +1, \ldots, +1)$. The smallest realization of (A.1) is $2^{[D/2]} \times 2^{[D/2]}$-dimensional ($[D/2]$ denotes the integer part of $D/2$). One often uses antisymmetrized products

$$\Gamma^{A_1\cdots A_p} \equiv \Gamma^{[A_1} \cdots \Gamma^{A_p]} \equiv \frac{1}{p!}\left(\Gamma^{A_1} \cdots \Gamma^{A_p} \pm \text{permutations}\right) , \tag{A.2}$$

with + (−) sign for even (odd) permutations.

The generators of $SO(1, D-1)$ in the spinor representation are

$$T^{AB} \equiv -\frac{i}{2}\Gamma^{AB} \equiv -\frac{i}{4}[\Gamma^A, \Gamma^B] . \tag{A.3}$$

Spinor representations are necessary to describe space-time fermions. Strictly speaking, when discussing spinors we should go to the covering group, the spin group. We will not make this distinction here but it is always implied.

Exercise A.1: Verify that $T^{AB} \equiv -\frac{i}{2}\Gamma^{AB}$ are generators of $SO(1,D-1)$ in the spinor representation, i.e.

$$i[T^{AB}, T^{CD}] = \eta^{AC}T^{BD} - \eta^{AD}T^{BC} - \eta^{BC}T^{AD} + \eta^{BD}T^{AC} \ . \tag{A.4}$$

Dirac spinors have then dimension $2^{[D/2]}$. For D even the Dirac representation is reducible since there exists a matrix that commutes with all generators. This is

$$\Gamma_{D+1} \equiv e^{-i\pi(D-2)/4}\Gamma^0 \dots \Gamma^{D-1} \ . \tag{A.5}$$

For D odd, $\Gamma_{D+1} \propto \mathbb{1}$.

Exercise A.2: Show that $\Gamma_{D+1}^2 = \mathbb{1}$, $\{\Gamma_{D+1}, \Gamma^A\} = 0$, and $[\Gamma_{D+1}, \Gamma^{AB}] = 0$.

With the help of Γ_{D+1} we can define the irreducible inequivalent *Weyl representations*: if ψ is a Dirac spinor, the left and right Weyl spinors are

$$\psi_L = \frac{1}{2}(1 - \Gamma_{D+1})\psi \, , \qquad \psi_R = \frac{1}{2}(1 + \Gamma_{D+1})\psi \ . \tag{A.6}$$

Note that $\Gamma_{D+1}\psi_R = \psi_R$ and $\Gamma_{D+1}\psi_L = -\psi_L$.

Dirac and Weyl spinors are complex but in some cases a *Majorana condition* of the form $\psi^* = B\psi$ with B a matrix such that $BB^* = \mathbb{1}$ is consistent with the Lorentz transformations $\delta\psi = i\omega_{MN}T^{MN}\psi$, i.e. B must satisfy $T^{*MN} = -BT^{MN}B^{-1}$. The Majorana condition is allowed for $D = 0, 1, 2, 3, 4$ mod 8. Majorana-Weyl spinors can be shown to exist only in $D = 2$ mod 8 [6].

$SO(D)$ spinors have analogous properties. For D even, there are two inequivalent irreducible Weyl representations of dimension $2^{D/2-1}$. A Majorana-Weyl condition can be imposed only for $D = 0$ mod 8.

A.2: Differential geometry

We use $A, B, \dots$ to denote flat tangent indices (raised and lowered with η^{AB} and η_{AB}) which are related to the curved indices $M, N, \dots$ (raised and lowered with G^{MN} and G_{MN}) via the D-bein: e.g. $\Gamma^A = e^A_M\Gamma^M$ and the inverse D-bein, e.g. $\Gamma^M = e^M_A\Gamma^A$, where $G_{MN} = e^A_M e^B_N \eta_{AB}$ and $e^A_M e^M_B = \delta^A_B$, $e^A_M e^N_A = \delta^N_M$, $\eta^{AB}\eta_{BC} = \delta^A_C$. The Γ^M satisfy $\{\Gamma^M, \Gamma^N\} = 2G^{MN}$.

A Riemannian connection Γ^P_{MN} is defined by imposing

$$\begin{aligned} \nabla_P G_{MN} &\equiv \partial_p G_{MN} - \Gamma^Q_{PM}G_{QN} - \Gamma^Q_{PN}G_{MQ} = 0 \qquad \text{(metricity)} \\ \Gamma^P_{MN} &= \Gamma^P_{NM} \qquad \text{(no torsion)} \ . \end{aligned} \tag{A.7}$$

One finds for the *Christoffel symbols*

$$\Gamma^P_{MN} = \frac{1}{2}G^{PQ}\left(\partial_M G_{QN} + \partial_N G_{MQ} - \partial_Q G_{MN}\right) \ . \tag{A.8}$$

The *Riemann tensor* is

$$[\nabla_M, \nabla_N]V_P = -R_{MNP}{}^Q V_Q \ . \tag{A.9}$$

The *Ricci tensor* and the *Ricci scalar* are $R_{MN} = G^{PQ}R_{MPNQ}$ and $R = G^{MN}R_{MN}$. The *spin connection* is defined via the condition

$$\nabla_M e^A_N = \partial_M e^A_N - \Gamma^P_{MN}e^A_P + \omega_{MB}{}^A e^B_M = 0 \tag{A.10}$$

which leads to the following explicit expression for its components

$$\begin{aligned}\omega_M^{AB} &= \frac{1}{2}e^{NA}(\partial_M e_N^n - \partial_N e_M^b) - \frac{1}{2}e^{NB}(\partial_M e_N^a - \partial_N e_M^A) \\ &\quad -\frac{1}{2}e^{RA}e^{SB}(\partial_R e_{SC} - \partial_S e_{RC})e_M^C) \ . \end{aligned} \tag{A.11}$$

In terms of ω_M^{AB} the components of the Lie-algebra valued curvature 2-form are

$${R_{MN}}^{AB} = e^{AP}e^{BQ}R_{MNPQ} = \partial_M\omega_N^{AB} - \partial_N\omega_M^{AB} + \omega_M^{AC}{\omega_{NC}}^B - \omega_N^{AC}{\omega_{MC}}^B \ . \tag{A.12}$$

The covariant derivative, acting on an object with only tangent-space indices, is generically

$$\nabla_M = \partial_M + \frac{i}{2}\omega_M^{AB}T_{AB} \ , \tag{A.13}$$

where T_{AB} is a generator of the tangent space group $SO(1, D-1)$. For example, $i{(T_{AB})_C}^D = \eta_{AC}\delta_B^D - \eta_{BC}\delta_A^D$ for vectors and $iT_{AB} = \frac{1}{2}\Gamma_{AB}$ for spinors (spinor indices are suppressed).

Under infinitesimal parallel transport a vector V changes as $\delta V^M = dx^N\nabla_N V^M = dx^N(\partial_N V^M + \Gamma^M_{NR}V^R)$. When V is transported around an infinitesimal loop in the (M, N)-plane with area $\delta a^{MN} = -\delta a^{NM}$ it changes by the amount

$$\delta V^P = -\frac{1}{2}\delta a^{MN}{R_{MN}}^P{}_Q V^Q \, . \tag{A.14}$$

Notice that under parallel transport the length $|V|$ remains constant since $|V|^2 = V^MV^NG_{MN}$ and $\nabla_P G_{MN} = 0$. The generalization to the parallel transport of tensors and spinors is obvious.

Appendix B: First Chern class of hypersurfaces of $\mathbb{P}^n$

This Appendix is adopted from [27].

Let $X = \{z \in \mathbb{P}^n; f(z) = 0\}$, f a homogeneous polynomial of degree d, be a non-singular hypersurface in $\mathbb{P}^n$. From (3.70) we know that $c_1(X)$ can be expressed by any choice of volume element on X. As volume element we will use the pull-back of the $(n-1)$-st power of the Kähler form on $\mathbb{P}^n$. We will first compute this in general and will then use the Fubini-Study metric on $\mathbb{P}^n$. It suffices to do the calculation on the subset

$$U_0 \cap \{\partial_n f \neq 0\} \cap X \ . \tag{B.1}$$

Given that $\omega = ig_{i\bar{\jmath}}dz^i \wedge dz^{\bar{\jmath}}$ we compute (z^i are the inhomogeneous coordinates on U_0)

$$\omega^{n-1} = (ig_{i\bar{\jmath}}dz^i \wedge d\bar{z}^j)^{n-1} = i^{n-1}\sum_{i,j=1}^{n}(-1)^{i+j}\det(m_{ij})(\cdots\widehat{i}\cdots\widehat{\bar{\jmath}}\cdots) \ , \tag{B.2}$$

where m_{ij} is the (i, j)-minor of the metric $g_{i\bar{\jmath}}$. The notation $(\cdots\widehat{i}\cdots\widehat{\bar{\jmath}}\cdots)$ means that the hatted factors are missing in the product $dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$. In the next step we split the above sum according to how many powers of dz^n appear. We get

$$\begin{aligned} i^n\omega^{n-1} &= \sum_{i,j=1}^{n-1}(-1)^{i+j}\det(m_{ij})(\cdots\widehat{i}\cdots\widehat{\bar{\jmath}}\cdots) \\ &+ \sum_{i=1}^{n-1}(-1)^{n+i}\det(m_{in})(\cdots\widehat{i}\cdots\widehat{\bar{n}}) + \sum_{j=1}^{n-1}(-1)^{n+j}\det(m_{hj})(\cdots\widehat{\bar{\jmath}}\cdots\widehat{n}\cdot) \\ &+ \det(m_{nn})(\cdots\widehat{n}\,\widehat{\bar{n}}) \ . \end{aligned} \tag{B.3}$$

We now replace dz^n via the hypersurface constraint:

$$df = \sum_{i=1}^{n-1} \frac{\partial f}{\partial z^i} dz^i + \frac{\partial f}{\partial z^n} dz^n \quad \Rightarrow \quad dz^n = -\left(\frac{\partial f}{\partial z^n}\right)^{-1} \sum_{i=1}^{n-1} \frac{\partial f}{\partial z^i} dz^i \ . \tag{B.4}$$

Using this in (B.3), we find

$$(i)^n \omega^{n-1} = \left|\frac{\partial f}{\partial z^n}\right|^{-2} \sum_{i,j=1}^{n} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}} (-1)^{i+j} \det(m_{ij})(dz^1 \wedge \cdots \wedge d\bar{z}^{n-1}) \ . \tag{B.5}$$

Next we need the identity

$$g^{i\bar{j}} \equiv (g^{-1})_{i\bar{j}} = (-1)^{i+j} \det(m_{ij})(\det g)^{-1} \ . \tag{B.6}$$

Using this in (B.4), we obtain

$$(-i)^n \omega^{n-1} = (\det g) \left|\frac{\partial f}{\partial z^n}\right|^{-2} \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}} (dz^1 \wedge \cdots \wedge d\bar{z}^n) \ . \tag{B.7}$$

We now specify to the Fubini-Study metric, for which

$$g^{i\bar{j}} = (1 + |z|^2)(\delta_{ij} + z^i \bar{z}^j) \ . \tag{B.8}$$

It follows that

$$\sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}} = (1 + |z|^2) \left(\sum_{i=1}^{n} \left|\frac{\partial f}{\partial z^i}\right|^2 + \sum_{i,j=1}^{n} z^i \frac{\partial f}{\partial z^i} \overline{z^j \frac{\partial f}{\partial z^j}} \right) \ . \tag{B.9}$$

Now, since f vanishes on X and since it is a homogeneous function of degree d, on X we get

$$0 = d \cdot f = \frac{\partial f}{\partial z^0} + \sum_{i=1}^{n} z^i \frac{\partial f}{\partial z^i} \ , \tag{B.10}$$

and therefore

$$\sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}} = (1 + |z|^2) \left(\sum_{i=0}^{n} \left|\frac{\partial f}{\partial z^i}\right|^2 \right) \ . \tag{B.11}$$

Because the determinant of the metric is (cf. (3.28))

$$\det(g_{i\bar{j}}) = \frac{1}{(1 + |z|^2)^{n+1}} \ , \tag{B.12}$$

we find

$$(i)^n \omega^{n-1} = \left|\frac{\partial f}{\partial z^n}\right|^{-2} \frac{\sum_{i=0}^{n} \left|\frac{\partial f}{\partial z^i}\right|^2}{(|z|^2)^n} \tag{B.13}$$

where now $|z|^2 = \sum_{i=0}^{n} |z^i|^2$. If we set

$$\psi = \log \left(\frac{\sum_{i=0}^{n} |\partial_i f|^2}{|z|^{2d-2}} \right) \ , \tag{B.14}$$

which is a globally defined function, i.e. it has a unique value on all overlaps, we can write

$$\begin{aligned} \partial\bar{\partial} \log \omega^{n-1} &= \partial\bar{\partial} \log \frac{\sum \left|\frac{\partial f}{\partial z_i}\right|^2}{(|z|^2)^n} - \partial\bar{\partial} \log \left|\frac{\partial f}{\partial z_n}\right|^2 \\ &= \partial\bar{\partial} \log e^{\psi} (|z|^2)^{d-n-1} \\ &= \partial\bar{\partial}\psi + i(n - d + 1)\omega \ . \end{aligned} \tag{B.15}$$

Recall that this is valid on the subset specified in (B.1), in particular that this expression is to be evaluated on the hypersurface $f(z) = 0$. Comparing this to (3.70) we realize that we have shown that

$$2\pi c_1(X) = (n + 1 - d)[\omega] \ . \tag{B.16}$$

Appendix C: Partition function of type II strings on $\mathrm{T}^{10-d}/\mathbb{Z}_N$

The starting point is the partition function for the ten-dimensional type II strings that can be written as the product of a bosonic $\mathcal{Z}_B$ and a fermionic $\mathcal{Z}_F$ contribution [4, 5, 6]. Up to normalization:

$$\begin{aligned}
\mathcal{Z}(\tau,\bar\tau) &= \mathcal{Z}_B(\tau,\bar\tau)\,\mathcal{Z}_F(\tau,\bar\tau)\\
\mathcal{Z}_B(\tau,\bar\tau) &= \left(\frac{1}{\sqrt{\tau_2}\,\eta\,\bar\eta}\right)^8 \\
\mathcal{Z}_F(\tau,\bar\tau) &= \frac{1}{4}\left\{\frac{\vartheta^4[{0\atop 0}]}{\eta^4}-\frac{\vartheta^4[{0\atop \frac12}]}{\eta^4}-\frac{\vartheta^4[{\frac12\atop 0}]}{\eta^4}+\frac{\vartheta^4[{\frac12\atop \frac12}]}{\eta^4}\right\}\times\\
&\qquad\left\{\frac{\bar\vartheta^4[{0\atop 0}]}{\bar\eta^4}-\frac{\bar\vartheta^4[{0\atop \frac12}]}{\bar\eta^4}-\frac{\bar\vartheta^4[{\frac12\atop 0}]}{\bar\eta^4}\pm\frac{\bar\vartheta^4[{\frac12\atop \frac12}]}{\bar\eta^4}\right\}\ ,
\end{aligned}\tag{C.1}$$

where $\eta(\tau)$ is the Dedekind function defined in (4.27) and the Jacobi theta functions are

$$\vartheta[{\delta\atop\varphi}](\tau)=\sum_n q^{\frac12(n+\delta)^2}\,e^{2i\pi(n+\delta)\varphi}\quad ;\quad q=e^{2i\pi\tau}\ .\tag{C.2}$$

The theta functions also have the product form (4.42) given in section 4.3. In the following we will not write explicitly that ϑ and η are functions of τ.

Depending on the sign in the last term of the right-moving piece of $\mathcal{Z}_F$ we have type IIB (+ sign) or IIA (− sign) strings. In the following we consider type IIB so that $\mathcal{Z}_F(\tau,\bar\tau)=|\mathcal{Z}_F(\tau)|^2$. The left-moving $\mathcal{Z}_F(\tau)$ can be written as:

$$\mathcal{Z}_F(\tau)=\frac12\sum_{\alpha,\beta=0,\frac12} s_{\alpha\beta}\,\frac{\vartheta^4[{\alpha\atop\beta}]}{\eta^4}\ .\tag{C.3}$$

The $s_{\alpha\beta}$ are the spin structure coefficients. Modular invariance requires $s_{0\frac12}=s_{\frac12 0}=-s_{00}$. This can be checked using the transformation properties:

$$\begin{aligned}
\mathcal{T}\ &:\ \tau\to\tau+1\quad ;\quad \eta\to e^{\frac{i\pi}{12}}\eta\quad ;\quad \vartheta[{\alpha\atop\beta}]\to e^{-i\pi(\alpha^2-\alpha)}\vartheta[{\alpha\atop \alpha+\beta-\frac12}]\ ,\\
\mathcal{S}\ &:\ \tau\to -1/\tau\quad ;\quad \eta\to(-i\tau)^{\frac12}\eta\quad ;\quad \vartheta[{\alpha\atop\beta}]\to(-i\tau)^{\frac12}e^{2i\pi\alpha\beta}\vartheta[{\beta\atop -\alpha}]\ .
\end{aligned}\tag{C.4}$$

We take $s_{00}=1$ and choose $s_{\frac12\frac12}$ equal to s_{00} so that the GSO projections in the NS and R sectors turn out the same as we explain below.

The NS sector corresponds to $\alpha=0$. Using (C.2) we can write

$$\frac12\left\{\frac{\vartheta^4[{0\atop 0}]}{\eta^4}-\frac{\vartheta^4[{0\atop \frac12}]}{\eta^4}\right\}=\frac{1}{\eta^4}\sum_{r_a\in\mathbb{Z}} q^{\frac12 r^2}\left[\frac{1-e^{i\pi(r_0+r_1+r_2+r_3)}}{2}\right]\ .\tag{C.5}$$

This shows that left-moving fermionic degrees of freedom of a given NS state depend on a vector r with four integer entries. This is an $SO(8)$ weight in the scalar or vector class. Similarly, for the R sector with $\alpha=\frac12$ we have

$$-\frac12\left\{\frac{\vartheta^4[{\frac12\atop 0}]}{\eta^4}-\frac{\vartheta^4[{\frac12\atop \frac12}]}{\eta^4}\right\}=-\frac{1}{\eta^4}\sum_{r_a\in\mathbb{Z}+\frac12} q^{\frac12 r^2}\left[\frac{1-e^{i\pi(r_0+r_1+r_2+r_3)}}{2}\right]\ .\tag{C.6}$$

Now r has half-integer entries so that it corresponds to an $SO(8)$ spinor weight. Here we are actually exchanging the light-cone world-sheet fermions by four free bosons that have momentum r in the $SO(8)$ weight lattice. This equivalence between fermions and bosons is in fact seen in (C.5) and (C.6) when we write the left-hand-side using (4.42). Furthermore, because we have included both $\beta=0$ and $\beta=\frac{1}{2}$, only states with $\sum r_a =$ odd do appear. This is the GSO projection. For instance, the tachyon $r=0$ in the NS sector is eliminated from the spectrum. In the R sector one of the $SO(8)$ spinor representations with $r^2=1$ is also absent. For the right-moving piece we obtain completely analogous results in terms of an $SO(8)$ weight denoted p.

Let us now discuss the partition function for the orbifold that has the form (4.39). Each term $\mathcal{Z}(\theta^k,\theta^\ell)$ can be written as the product of bosonic and fermionic pieces. The bosonic piece is

$$\mathcal{Z}_B(\theta^k,\theta^\ell) = \left(\frac{1}{\sqrt{\tau_2}\,\eta\,\bar\eta}\right)^{d-2} \chi(\theta^k,\theta^\ell) \left|\prod_{j=1}^{5-\frac{d}{2}} \frac{\eta}{\vartheta\left[\begin{smallmatrix}\frac12+kv_j\\ \frac12+\ell v_j\end{smallmatrix}\right]}\right|^2 , \tag{C.7}$$

where $\chi(\theta^k,\theta^\ell)$ is the number of simultaneous fixed points of θ^k and θ^ℓ. The first term is the contribution of the non-compact coordinates ($d-2$ in the light-cone gauge), whereas the second term comes from the $(10-d)$ compact coordinates as we have seen in section (4.3). We are assuming d even.

For the fermionic piece we start with the untwisted sector. The insertion of θ^ℓ in the trace leads to

$$\mathcal{Z}_F(\mathbb{1},\theta^\ell) = \frac14 \left|\sum_{\alpha,\beta=0,\frac12} s_{\alpha\beta}(0,\ell)\,\frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}{\eta}\prod_{j=1}^{3}\frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta+\ell v_j\end{smallmatrix}\right]}{\eta}\right|^2 , \tag{C.8}$$

where $s_{\alpha\beta}(0,\ell)=s_{\alpha\beta}(0,0)$ are the spin structures in (C.3). We have specialized to $d=4$, for other cases simply set $v_j=0$ for $j>5-\frac{d}{2}$. To derive the remaining $\mathcal{Z}_F(\theta^k,\theta^\ell)$ we use modular transformations. In the end we obtain

$$\mathcal{Z}_F(\theta^k,\theta^\ell) = \frac14 \left|\sum_{\alpha,\beta=0,\frac12} s_{\alpha\beta}(k,\ell)\,\frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]}{\eta}\prod_{j=1}^{3}\frac{\vartheta\left[\begin{smallmatrix}\alpha+kv_j\\ \beta+\ell v_j\end{smallmatrix}\right]}{\eta}\right|^2 . \tag{C.9}$$

Modular invariance imposes relations among the spin structure coefficients. We find:

$$s_{00}(k,\ell) = -s_{\frac12 0}(k,\ell) = 1 \quad ; \quad s_{0\frac12}(k,\ell) = -s_{\frac12\frac12}(k,\ell) = -e^{-i\pi k(v_1+v_2+v_3)} . \tag{C.10}$$

Setting $k=N$ then gives a further condition on the twist vector, namely:

$$N(v_1+v_2+v_3) = 0 \bmod 2 . \tag{C.11}$$

Notice that all twists in Table 2 do satisfy this condition.

Exercise C.1 : Use (C.4) to show that (C.9) has the correct modular transformations, i.e. $\mathcal{Z}_F(\theta^k,\theta^\ell)$ transforms into $\mathcal{Z}_F(\theta^k,\theta^{k+\ell})$ under $\mathcal{T}$ and into $\mathcal{Z}_F(\theta^\ell,\theta^{-k})$ under $\mathcal{S}$.

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Index