2 Kaluza-Klein Fundamentals

Exercise 2.1: Perform the dimensional reduction of:

- Maxwell electrodynamics.

$$S_1 = -\frac{1}{4}\int d^{4+n}x\, F_{MN}F^{MN} \quad , \quad F_{MN} = \partial_M A_N - \partial_N A_M \ .$$

- Action for a 2-form gauge field B_{MN}.

$$S_2 = -\frac{1}{12}\int d^{4+n}x\, H_{MNP}H^{MNP} \quad , \quad H_{MNP} = \partial_M B_{NP} + \text{cyclic} \ .$$

Exercise 2.2: Use

$$R_{MN}{}^{AB} = e^{AP}e^{BQ}R_{MNPQ} = \partial_M\omega_N^{AB} - \partial_N\omega_M^{AB} + \omega_M^{AC}\,\omega_{NC}{}^B - \omega_N^{AC}\,\omega_{MC}{}^B \ ,$$

to verify

$$[\nabla_M, \nabla_N]\epsilon = \frac{1}{4}R_{MN}{}^{AB}\Gamma_{AB}\epsilon = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\epsilon = 0 \ .$$

Exercise 2.3: Consider the manifold $S^1 \otimes \mathbb{R}^n$ endowed with the metric

$$ds^2 = R^2 d\theta^2 + (dx^i + \Omega^i{}_j x^j d\theta)^2 \ ,$$

where $\Omega^i{}_j$ is a constant anti-symmetric matrix, i.e. a generator of the rotation group $SO(n)$ and R is the radius of S^1. Show that this metric has vanishing curvature but that nevertheless a vector, when parallel transported around the circle, is rotated by an element of $SO(n)$.

Exercise 2.4: Use simple group theory to work out the condition on the holonomy group of seven- and eight-dimensional manifolds which gives the minimal amount of supersymmetry if one compactifies eleven-dimensional supergravity to four or three dimensions or ten-dimensional supergravity to $d = 3$ and $d = 2$, respectively.