The water molecule, for example, has a rather strong dipole moment. The electric fields that result from this moment are responsible for some of the important properties of water. For many molecules, for example CO_2, the dipole moment vanishes because of the symmetry of the molecule. For them we should expand still more accurately, obtaining another term in the potential which decreases as $1/R^3$, and which is called a quadrupole potential. We will discuss such cases later.

6–6 The fields of charged conductors

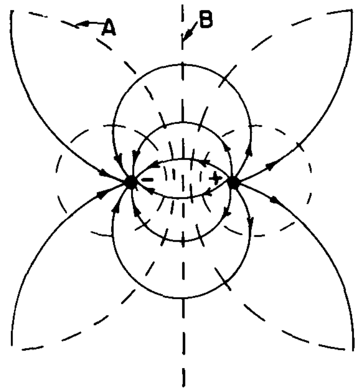

Fig. 6–8. The field lines and equipotentials for two point charges.

We have now finished with the examples we wish to cover of situations in which the charge distributions is known from the start. It has been a problem without serious complications, involving at most some integrations. We turn now to an entirely new kind of problem, the determination of the fields near charged conductors.

Suppose that we have a situation in which a total charge Q is placed on an arbitrary conductor. Now we will not be able to say exactly where the charges are. They will spread out in some way on the surface. How can we know how the charges have distributed themselves on the surface? They must distribute themselves so that the potential of the surface is constant. If the surface were not an equipotential, there would be an electric field inside the conductor, and the charges would keep moving until it became zero. The general problem of this kind can be solved in the following way. We guess at a distribution of charge and calculate the potential. If the potential turns out to be constant everywhere on the surface, the problem is finished. If the surface is not an equipotential, we have guessed the wrong distribution of charges, and should guess again—hopefully with an improved guess! This can go on forever, unless we are judicious about the successive guesses.

The question of how to guess at the distribution is mathematically difficult. Nature, of course, has time to do it; the charges push and pull until they all balance themselves. When we try to solve the problem, however, it takes us so long to make each trial that that method is very tedious With an arbitrary group of conductors and charges the problem can be very complicated, and in general it cannot be solved without rather elaborate numerical methods. Such numerical computations, these days, are set up on a computing machine that will do the work for us, once we have told it how to proceed.

On the other hand, there are a lot of little practical cases where it would be nice to be able to find the answer by some more direct method—without having to write a program for a computer. Fortunately, there are a number of cases where the answer can be obtained by squeezing it out of Nature by some trick or other. The first trick we will describe involves making use of solutions we have already obtained for situations in which charges have specified locations.

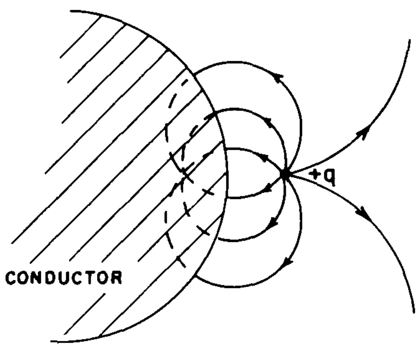

Fig. 6–9. The field outside a conductor shaped like the equipotential A of Fig. 6–8.

6–7 The method of images

We have solved, for example, the field of two point charges. Figure 6–8 shows some of the field lines and equipotential surfaces we obtained by the computations in Chapter 5. Now consider the equipotential surface marked A. Suppose we were to shape a thin sheet of metal so that it just fits this surface. If we place it right at the surface and adjust its potential to the proper value, no one would ever know it was there, because nothing would be changed.

But notice! We have really solved a *new* problem. We have a situation in which the surface of a curved conductor with a given potential is placed near a point charge. If the metal sheet we placed at the equipotential surface eventually closes on itself (or, in practice, if it goes far enough) we have the kind of situation considered in Section 5–10, in which our space is divided into two regions, one inside and one outside a closed conducting shell. We found there that the fields in the two regions are quite independent of each other. So we would have the same fields outside our curved conductor no matter what is inside. We can even fill up the whole inside with conducting material. We have found, therefore, the fields for the arrangement of Fig. 6–9. In the space outside the conductor the field is just like that of two point charges, as in Fig. 6–8. Inside the conductor, it is zero Also—as it must be—the electric field just outside the conductor is normal to the surface.

Thus we can compute the fields in Fig. 6–9 by computing the field due to q and to an imaginary point charge $-q$ at a suitable point. The point charge we "imagine" existing behind the conducting surface is called an *image charge.*

In books you can find long lists of solutions for hyperbolic-shaped conductors and other complicated looking things, and you wonder how anyone ever solved these terrible shapes. They were solved backwards! Someone solved a simple problem with given charges. He then saw that some equipotential surface showed up in a new shape, and he wrote a paper in which he pointed out that the field outside that particular shape can be described in a certain way.

6–8 A point charge near a conducting plane

As the simplest application of the use of this method, let's make use of the plane equipotential surface B of Fig. 6–8. With it, we can solve the problem of a charge in front of a conducting sheet. We just cross out the left-hand half of the picture. The field lines for our solution are shown in Fig. 6–10. Notice that the plane, since it was halfway between the two charges, has zero potential. We have solved the problem of a positive charge next to a grounded conducting sheet.

We have now solved for the total field, but what about the *real* charges that are responsible for it? There are, in addition to our positive point charge, some induced negative charges on the conducting sheet that have been attracted by the positive charge (from large distances away). Now suppose that for some technical reason—or out of curiosity—you would like to know how the negative charges are distributed on the surface. You can find the surface charge density by using the result we worked out in Section 5–6 with Gauss' theorem. The normal com-

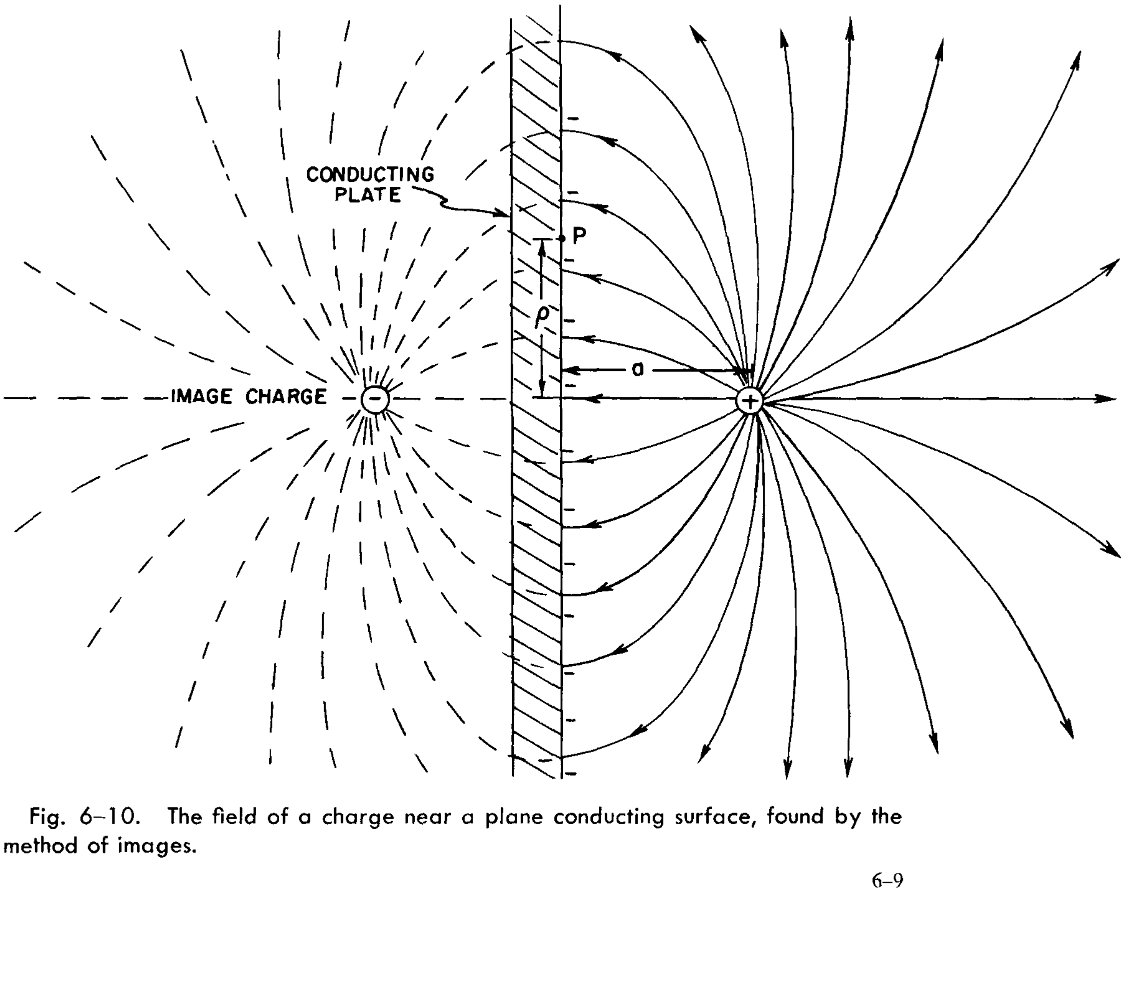

Fig. 6–10. The field of a charge near a plane conducting surface, found by the method of images.

ponent of the electric field just outside a conductor is equal to the density of surface charge σ divided by ϵ_0. We can obtain the density of charge at any point on the surface by working backwards from the normal component of the electric field at the surface. We know that, because we know the field everywhere.

Consider a point on the surface at the distance ρ from the point directly beneath the positive charge (Fig. 6–10). The electric field at this point is normal to the surface and is directed into it. The component normal to the surface of the field from the *positive* point charge is

$$E_{n+} = -\frac{1}{4\pi\epsilon_0}\frac{aq}{(a^2+\rho^2)^{3/2}} \tag{6.28}$$

To this we must add the electric field produced by the negative image charge. That just doubles the normal component (and cancels all others), so the charge density σ at any point on the surface is

$$\sigma(\rho) = \epsilon_0 E(\rho) = -\frac{2aq}{4\pi(a^2+\rho^2)^{3/2}}. \tag{6.29}$$

An interesting check on our work is to integrate σ over the whole surface. We find that the total induced charge is $-q$, as it should be.

One further question: Is there a force on the point charge? Yes, because there is an attraction from the induced negative surface charge on the plate. Now that we know what the surface charges are (from Eq. (6.29)), we could compute the force on our positive point charge by an integral. But we also know that the force acting on the positive charge is exactly the same as it *would be* with the negative image charge instead of the plate, because the fields in the neighborhood are the same in both cases. The point charge feels a force toward the plate whose magnitude is

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2a)^2}. \tag{6.30}$$

We have found the force much more easily than by integrating over all the negative charges.

6–9 A point charge near a conducting sphere

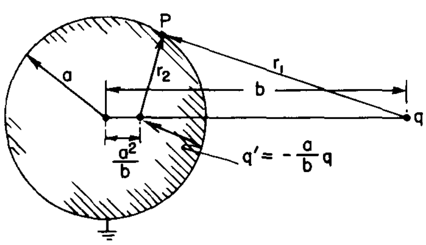

Fig. 6–11. The point charge q induces charges on a grounded conducting sphere whose fields are those of an image charge q' placed at the point shown.

What other surfaces besides a plane have a simple solution? The next most simple shape is a sphere. Let's find the fields around a metal sphere which has a point charge q near it, as shown in Fig. 6–11. Now we must look for a simple physical situation which gives a sphere for an equipotential surface. If we look around at problems people have already solved, we find that someone has noticed that the field of two *unequal* point charges has an equipotential that is a sphere. Aha! If we choose the location of an image charge—and pick the right amount of charge—maybe we can make the equipotential surface fit our sphere. Indeed, it can be done with the following prescription.

Assume that you want the equipotential surface to be a sphere of radius a with its center at the distance b from the charge q. Put an image charge of strength $q' = -q(a/b)$ on the line from the charge to the center of the sphere, and at a distance a^2/b from the center. The sphere will be at zero potential.

The mathematical reason stems from the fact that a sphere is the locus of all points for which the distances from two points are in a constant ratio. Referring to Fig. 6–11, the potential at P from q and q' is proportional to

$$\frac{q}{r_1} + \frac{q'}{r_2}.$$

The potential will thus be zero at all points for which

$$\frac{q'}{r_2} = -\frac{q}{r_1} \quad \text{or} \quad \frac{r_2}{r_1} = -\frac{q'}{q}.$$

If we place q' at the distance a^2/b from the center, the ratio r_2/r_1 has the constant value a/b. Then if

$$\frac{q'}{q} = -\frac{a}{b}, \tag{6.31}$$

the sphere is an equipotential. Its potential is, in fact, zero.

What happens if we are interested in a sphere that is not at zero potential? That would be so only if its total charge happens accidentally to be q' Of course if it is grounded, the charges induced on it would have to be just that. But what if it is insulated, and we have put no charge on it? Or if we know that the total charge Q has been put on it? Or just that it has a given potential *not* equal to zero? All these questions are easily answered. We can always add a point charge q'' at the center of the sphere The sphere still remains an equipotential by superposition; only the magnitude of the potential will be changed.

If we have, for example, a conducting sphere which is initially uncharged and insulated from everything else, and we bring near to it the positive point charge q, the total charge of the sphere will remain zero. The solution is found by using an image charge q' as before, but, in addition, adding a charge q'' at the center of the sphere, choosing

$$q'' = -q' = \frac{a}{b}q. \tag{6.32}$$

The fields everywhere outside the sphere are given by the superposition of the fields of q, q', and q''. The problem is solved.

We can see now that there will be a force of attraction between the sphere and the point charge q. It is not zero even though there is no charge on the neutral sphere. Where does the attraction come from? When you bring a positive charge up to a conducting sphere, the positive charge attracts negative charges to the side closer to itself and leaves positive charges on the surface of the far side. The attraction by the negative charges exceeds the repulsion from the positive charges, there is a net attraction. We can find out how large the attraction is by computing the force on q in the field produced by q' and q''. The total force is the sum of the attractive force between q and a charge $q' = -(a/b)q$, at the distance $b - (a^2/b)$, and the repulsive force between q and a charge $q'' = +(a/b)q$ at the distance b.

Those who were entertained in childhood by the baking powder box which has on its label a picture of a baking powder box which has on its label a picture of a baking powder box which has . may be interested in the following problem. Two equal spheres, one with a total charge of $+Q$ and the other with a total charge of $-Q$, are placed at some distance from each other. What is the force between them? The problem can be solved with an infinite number of images. One first approximates each sphere by a charge at its center. These charges will have image charges in the other sphere. The image charges will have images, etc , etc , etc The solution is like the picture on the box of baking powder—and it converges pretty fast.

6–10 Condensers; parallel plates

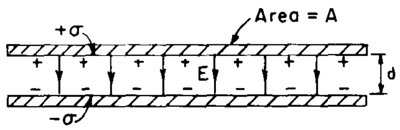

Fig. 6–12 A parallel-plate condenser.

We take up now another kind of a problem involving conductors. Consider two large metal plates which are parallel to each other and separated by a distance small compared with their width. Let's suppose that equal and opposite charges have been put on the plates. The charges on each plate will be attracted by the charges on the other plate, and the charges will spread out uniformly on the inner surfaces of the plates. The plates will have surface charge densities $+\sigma$ and $-\sigma$, respectively, as in Fig. 6–12. From Chapter 5 we know that the field between the plates is σ/ϵ_0, and that the field outside the plates is zero. The plates will have different potentials ϕ_1 and ϕ_2. For convenience we will call the difference V; it is often called the "voltage":

$$\phi_1 - \phi_2 = V.$$

(You will find that sometimes people use V for the potential, but we have chosen to use ϕ.)