Introduction to RIEMANN SURFACES

by

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PREFACE

The reawakening of interest in the subject of Riemann surfaces has brought with it the need for a textbook in English offering an introduction to the field. This book presents a self-contained, modern treatment of the fundamental concepts and basic theorems concerning Riemann surfaces. We assume that the reader is familiar with elementary complex function theory and with some real variables and algebra. Because we shall have to draw heavily from topology and Hilbert space theory, the reader will also find in this book an introduction to these fields, so that no previous knowledge of these subjects is required. This book is not meant to be a survey of the current work being done in the realm of Riemann surfaces, but rather is a modern presentation of the classical theory which will prepare the reader for further study in this and related fields.

Anyone writing a book on Riemann surface theory would certainly be influenced by the magnificient work of Professor Hermann Weyl in his *Idee der Riemannschen Fläche*, which laid the foundations for the theory of abstract Riemann surfaces. I am particularly indebted to this work, for it was there that I had my own introduction to the subject. I have also been very strongly influenced by the lectures on Riemann surfaces delivered by Professor Lars V. Ahlfors at Harvard University in 1948.

The original idea of writing this book came from Dr. L. Geller, who helped lay out the general plan and collaborated in writing Chapters 6 and 7. I am deeply indebted to him both for his help and for his enthusiasm. I wish to express my gratitude to Professor Maxwell Rosenlicht, who contributed numerous suggestions for making the proofs of many theorems more elegant, especially in the chapters on combinatorial topology and abelian integrals. My sincere thanks also go to the many other people who read the manuscript and offered constructive suggestions for improving it.

To find time to write such a book is always a difficult problem, and I am grateful for the C. L. E. Moore Instructorship at the Massachusetts Institute of Technology from 1949 to 1951 and to the Summer scholarship at Northwestern University in 1952 which gave me the opportunity to devote myself to this task. I also received many valuable suggestions from the 1956 Summer Seminar Group sponsored by the National Science Foundation at the University of Kansas. I wish to thank Miss Vera Fisher for the excellent job of typing the manuscript, and Addison-Wesley Publishing Company for their friendly cooperation in the publication of the final work.

January 1957 G. S.

CONTENTS

CHAPTER 1

INTRODUCTION

1–1 Algebraic functions and Riemann surfaces. A student in the theory of functions of a complex variable usually first encounters the notion of a Riemann surface in connection with the multiple-valued behavior of the function $w = \sqrt{z}$. In this book, we shall first regard a Riemann surface from a more abstract point of view. The aims of this introduction are to lead the reader over the bridge from the notion of several sheets covering the z-plane to the abstract definition, and to point out the goals of our study of Riemann surfaces and the routes we follow to attain these goals. The definitions made in the introduction will necessarily be vague and the arguments heuristic, but these will be set on a firm foundation in the later chapters.

An important part of the theory of functions of a complex variable is devoted to the study of algebraic functions and their integrals. An analytic function $w = w(z)$ is called an *algebraic function* if it satisfies a functional equation

$$a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0, \qquad a_0(z) \not\equiv 0,$$

in which the $a_i(z)$ are polynomials in z with complex numbers as coefficients. From this algebraic equation in w, we note that each value of z determines several values of w, so that w is a multiple-valued function of z. How the different values vary to form the continuous branches of $w(z)$ is one object of our investigation.

Moreover, a rational function of z and w is of the form

$$R(z, w) = \frac{b_0(z)w^m + b_1(z)w^{m-1} + \cdots + b_m(z)}{c_0(z)w^k + c_1(z)w^{k-1} + \cdots + c_k(z)}$$

where the $b_j(z)$ and $c_j(z)$ are polynomials in z with constant complex coefficients, and the denominator is not identically zero. We shall be interested in studying the function $F(z)$ defined by selecting one branch of an algebraic function $w(z)$ at z_0, a path from z_0 to z, and setting

$$F(z) = \int_{z_0}^{z} R(z, w(z))\, dz,$$

where the value of $w(z)$ is determined by analytic continuation along the path of integration from the fixed branch at z_0. In general, $F(z)$ is also a multiple-valued function of z. We shall find a system of canonical forms for these integrals so that any integral of this type can be transformed into a canonical form by a suitable change of variables. Then we shall study the canonical forms to learn more about the nature of these integrals.

Starting from a single function element of an algebraic function $w(z)$, we could use analytic continuation to piece together the whole function and in this way study its multiple-valuedness. In this book, however, we shall use Riemann's approach, in which one looks for a new surface (instead of the z-plane) on which to consider the algebraic function defined, and on which it is an ordinary single-valued function. It is this surface that we call a *Riemann surface.*

The simplest algebraic functions are those defined by an equation of the form $a_0(z)w + a_1(z) = 0$, where a_0 and a_1 are polynomials in z. In this case, $w = -a_1(z)/a_0(z)$ is a single-valued rational function of z; functions of this type are characterized by the condition that w be regular in the extended z-plane (z-sphere) except for a finite number of poles. If the poles occur at the points $b_1, b_2, \ldots, b_n$, then w may be expanded in partial fractions:

$$w = p(z) + h_1(z) + \cdots + h_n(z),$$

where

$$h_k(z) = \frac{c_{1,k}}{z - b_k} + \frac{c_{2,k}}{(z - b_k)^2} + \cdots + \frac{c_{m,k}}{(z - b_k)^m}$$

is the principal part of $w(z)$ at b_k and $p(z)$ is the polynomial in z which, to within a constant term, is the principal part of $w(z)$ at infinity. Any rational function $R(z, w)$ of z and this rational function w is also a rational function of z and has a partial-fraction expansion. Each integral

$$F(z) = \int_{z_0}^{z} R(z, w)\, dz$$

can be computed directly, yielding terms of the form $A \log (z - b)$, in addition to a rational function of z. Thus $F(z)$ is a multiple-valued function of z which changes value by $2\pi i A$ when z is continued around a small circle about any b which is a pole of $R(z, w)$ with nonzero residue A. Moreover, the change in value of $F(z)$ around any simple closed path is, by the residue theorem, $2\pi i$ times the sum of the residues of $R(z, w)$ at points interior to this path, so that the terms $A \log (z - b)$ account completely for the multiple-valuedness of $F(z)$. Thus we have some of the important properties of an algebraic function defined by a equation of degree 1 in w.

The next algebraic functions we shall consider are those defined by equations of degree 2 in w; that is, $a_0w^2 + a_1w + a_2 = 0$, where the $a_i = a_i(z)$ are polynomials in z, and $a_0 \not\equiv 0$. If we make the simple change of variable $\zeta = 2a_0w + a_1$, we obtain

$$\zeta^2 - p(z) = 0,$$

where $p(z) = a_1^2 - 4a_0a_2$ is a polynomial in z. For any fixed z, ζ is a single-valued function of w, and conversely; here, we shall study $\zeta(z)$ instead of $w(z)$. We shall do this by starting with $p(z)$ of degree 1 in z and letting the degree of p increase in going from one case to the next.

The algebraic function defined by $w^2 - z = 0$ is not single-valued in the extended z-plane. For, using polar coordinates $z = re^{i\theta}$, we have $w = \sqrt{r}\, e^{\frac{1}{2}i\theta}$. Starting at some point $r_0e^{i\theta_0}$, $r_0 \neq 0$, and continuing $w(z)$ along a closed path that winds once around the origin so that θ increases by 2π, $w(z)$ comes to the value $\sqrt{r_0}\, e^{\frac{1}{2}i(\theta_0+2\pi)} = -\sqrt{r_0}\, e^{\frac{1}{2}i\theta_0}$, which is just the negative of its original value. Continuation around this path once again leads back to the original value of $w(z)$. If we cut the extended z-plane along the positive real axis and restrict ourselves so as never to continue $w(z)$ over this cut, we get two single-valued branches of $w(z)$, namely, $w = \sqrt{r}\, e^{\frac{1}{2}i\theta}$, $0 \leq \theta < 2\pi$, and $w = \sqrt{r}\, e^{\frac{1}{2}i\theta}$, $2\pi \leq \theta < 4\pi$. To "build" the Riemann surface for $w(z)$, we take two replicas of the z-plane cut along the positive real axis and call them sheet I and sheet II. The cut on each sheet has two edges; label the edge of the first quadrant with a $+$ and the edge of the fourth quadrant with a $-$. Then attach the $+$ edge of the cut on I to the $-$ edge of the cut on II, and attach the $-$ edge of the cut on I to the $+$ edge of the cut on II. Thus, whenever we cross the cut, we pass from one sheet to the other.

Now the coordinate z determines a point in I and a point in II. It will be convenient to find a designation which will determine a single point on the Riemann surface. We associate to the point z on I the fixed value of $\sqrt{z}$ given by $\sqrt{r}\, e^{\frac{1}{2}i\theta}$, $0 \leq \theta < 2\pi$, and designate this point on I by $(z, \sqrt{z})$. Then, starting from $w = \sqrt{z}$, if we continue the function $w(z)$ defined by $w^2 - z = 0$ around a simple closed path about the origin, we cross the cut and pass into II, and when we return to the point in II having coordinate z, w has become $-\sqrt{z}$. We designate the point z on II by $(z, -\sqrt{z})$, which distinguishes it from $(z, \sqrt{z})$ on I. Thus each point of the Riemann surface may be considered as an ordered pair (z, w), where $w^2 - z = 0$, and two points (z_1, w_1) and (z_2, w_2) are identical on the Riemann surface if and only if $z_1 = z_2$ and $w_1(z) = w_2(z)$ about $z = z_1$. It is also clear that $w(z)$, satisfying $w^2 - z = 0$, is single-valued on the surface and assumes the value w at the point (z, w). In this case, there are two values of w for each base point z except $z = 0$ and $z = \infty$, which are branch points of $w = \sqrt{z}$.

Unfortunately, the two-sheeted surface we just constructed cannot be realized in our three-dimensional euclidean space as two sheets lying over the z-plane and attached crosswise along the given cuts, as will be readily apparent if we try to make it by cutting sheets of paper. It is this fact that lends an air of mystery to this surface, and which makes us suspicious and uncomfortable about Riemann surfaces in general. To dispel any such suspicion, we shall show that the two-sheeted surface can be mapped topologically onto a sphere.† Again we shall begin by imagining the surface as two sheets lying over the extended z-plane, each cut along the positive real axis. Using stereographic projection, we can consider the two sheets to be spheres cut along a meridian circle from the south pole to the north pole (Fig. 1–1) with each $+$ edge attached to the $-$ edge of the other sheet. Now pretend that the spheres are made of rubber and, by spreading the edges of the cuts, deform each sheet into a hemisphere. When each sheet

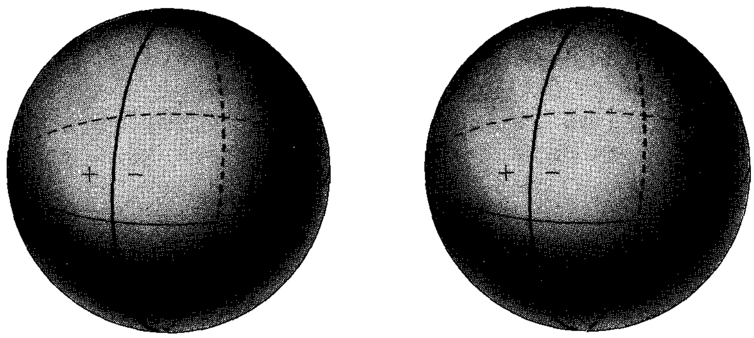


FIGURE 1–1.

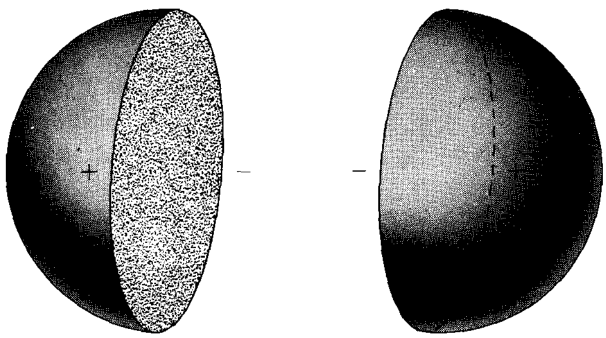


FIGURE 1–2.

†A mapping is called *topological* if it is continuous and one-to-one with a continuous inverse.

is rotated so that the openings of the hemispheres face each other (Fig. 1–2), the edges marked + and − face each other and the two hemispheres may be pasted together to give us a sphere. This mapping is carried out analytically if we take each point $(z, \sqrt{z})$ of the Riemann surface into the point $t = \sqrt{z}$ of the extended t-plane (t-sphere).

Now what can we say about the integrals

$$F(z) = \int_{z_0}^{z} R(z, \sqrt{z})\, dz,$$

where $R(z, w)$ is a rational function of z and w? If we consider this integral on the Riemann surface $(z, \sqrt{z})$ and map this surface onto the t-sphere by $t = \sqrt{z}$, our integral becomes

$$F(z) = \int_{\sqrt{z_0}}^{\sqrt{z}} R(t^2, t) 2t\, dt,$$

which is just the integral of a rational function of t. But this was treated in the first case, in which we saw that the only multiplicity arises from the residues of $2R(t^2, t)t$. Thus, $F(z)$ is a multiple-valued function on the Riemann surface of $w^2 - z = 0$, the multiplicity arising from the logarithmic singularities. Finally, in the z-plane itself, $F(z)$ has the additional two-valuedness due to the identification of the sheets.

The situation for $w^2 = a_0 z + a_1$ is essentially the same as that for $w^2 - z = 0$. Here we make the cut in the z-plane from $z = -a_1/a_0$ to $z = \infty$ instead of from 0 to ∞ and proceed as before. In fact, even the case $w^2 = a_0 z^2 + a_1 z + a_2$, $a_1^2 - 4a_0 a_2 \neq 0$, $a_0 \neq 0$, offers nothing essentially new, for, by factoring, we get $w^2 = a_0(z - r)(z - s)$, $r \neq s$. The two points $z = r$ and $z = s$ are branch points of this function, and we obtain two single-valued branches of $w = \sqrt{a_0(z - r)(z - s)}$ by cutting the z-plane along a curve joining r to s. Joining two replicas of the extended z-plane along this cut, we obtain a two-sheeted Riemann surface on which $w(z)$ is single-valued. It is clear that if the surface were made of rubber, it could be deformed continuously into that of $w^2 = z$ by moving r into 0 and s into ∞ and deforming the cut into the positive real axis. Thus this new surface may also be mapped topologically into a sphere. The mapping is executed analytically by first applying the linear fractional transformation $\tau = (z - r)/(z - s)$, which carries the z-plane in a one-to-one conformal manner onto the τ-plane with $r \to 0$ and $s \to \infty$. The two-sheeted Riemann surface over the z-plane maps onto a two-sheeted Riemann surface over the τ-plane, branched at $\tau = 0$ and $\tau = \infty$. Then $t = \sqrt{\tau}$ unwinds this Riemann surface and maps it onto the t-sphere as before.

We now consider the integral

$$\int_{z_0}^{z} R\left(z, \sqrt{a_0 z^2 + a_1 z + a_2}\right) dz$$

of a rational function of z and w, where $w^2 = a_0 z^2 + a_1 z + a_2$. Using the change of variables above, which maps the Riemann surface over the z-sphere onto the t-sphere, we have $t = \sqrt{(z-r)/(z-s)}$, and

$$\int_{z_0}^{z} R(z, w)\, dz = \int_{(z_0-r)/(z_0-s)}^{(z-r)/(z-s)} R\left(\frac{\tau s - r}{\tau - 1}, \sqrt{a_0 \tau}\, \frac{s-r}{\tau - 1}\right) \frac{r-s}{(\tau-1)^2}\, d\tau,$$

or

$$F(z) = \int_{\sqrt{(z_0-r)/(z_0-s)}}^{\sqrt{(z-r)/(z-s)}} R\left(\frac{t^2 s - r}{t^2 - 1}, \sqrt{a_0}\, t\, \frac{s-r}{t^2-1}\right) \frac{r-s}{(t^2-1)^2}\, 2t\, dt,$$

which is the integral of a rational function of t on the t-sphere. This integral is a multiple-valued function of $t = \sqrt{(z-r)/(z-s)}$ because of the logarithmic singularities corresponding to those poles for which the integrand has nonzero residue. Thus, as before, the multiple-valuedness of $F(z)$ in the z-plane arises from the logarithmic singularities of $F(z)$ and the two-valuedness of the map $z \rightarrow t$.

The picture changes significantly when we proceed to the case of the algebraic function defined by $w^2 = a(z - r_1)(z - r_2)(z - r_3)$, where r_1, r_2, r_3 are distinct. Again, to each z there correspond two values of w, one the negative of the other. We go from one to the other by continuing $w(z)$ over any closed path winding once around one of the roots r_1, r_2, r_3. For $w = \sqrt{a}\sqrt{z - r_1}\sqrt{z - r_2}\sqrt{z - r_3}$, and the factor $\sqrt{z - r_i}$ changes sign when arg $(z - r_i)$ changes by 2π. If we cut the z-plane from r_1 to r_2, we cannot wind around either r_1 or r_2 alone without crossing the cut. However, we could choose a path which winds around both r_1 and r_2 (see the dotted path in Fig. 1–3). But now both arg $(z - r_1)$ and arg $(z - r_2)$ change by 2π, both the factors $\sqrt{z - r_1}$ and $\sqrt{z - r_2}$ change sign, and there is no change in w. We next cut the z-plane from r_3 to ∞. This prevents us from winding around all three of the roots r_1, r_2, and r_3. Thus either branch of $w(z)$ is single-valued in the cut plane. If we now take two copies of the cut z-plane (Fig. 1–3 or 1–4) and connect them crosswise over the cuts as before, we obtain a two-sheeted Riemann surface on which $w^2 = a(z - r_1)(z - r_2)(z - r_3)$ is single-valued. Again the points on this surface can be designated by $(z, w(z))$, where the z determines a point on both sheets and $w(z)$ says on which sheet the point lies.

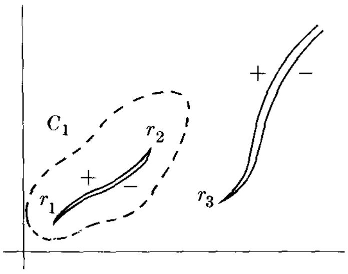
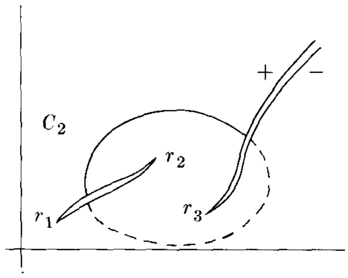


FIGURE 1-3.



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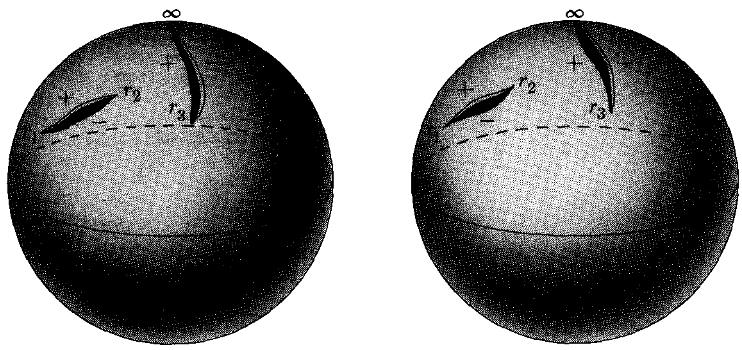

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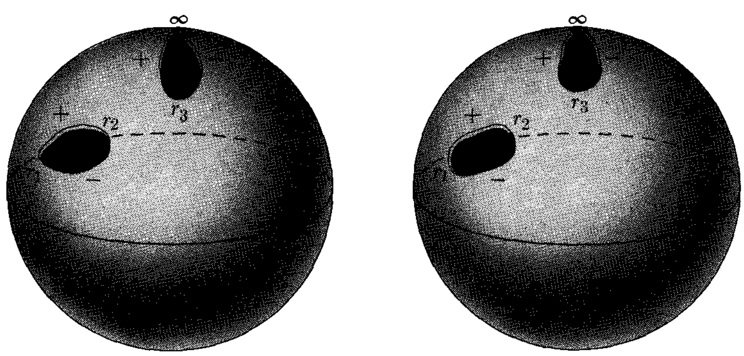

FIGURE 1-6.

This two-sheeted Riemann surface *cannot* be topologically mapped onto a sphere, but we now show that it can be mapped topologically onto a torus (doughnut). This can be seen by placing next to each other the two spheres cut between r_1 and r_2 and between r_3 and ∞. Each $+$ edge of a cut is to be attached to the $-$ edge of the corresponding cut on the other sphere (Fig. 1–5). Imagine that the spheres are made of rubber, and stretch each cut into a circular hole (Fig. 1–6). Then rotate the spheres until the holes face each other, and pull the edges of the cuts outward to make little tubes (Fig. 1–7). Notice that now the $+$ edges of the tubes on one sphere are opposite the $-$ edges of the tubes on the other sphere. Thus we may join together the ends of the tubes to form the surface in Fig. 1–8, which can be topologically mapped onto a torus (Fig. 1–9).

It is easy to see that the torus cannot be mapped onto a sphere topologically. For on the sphere, any closed curve can be deformed to a point and this property is preserved under topological mappings of the surface. On the torus, however, the meridian circles C_1 and the latitude circles C_2

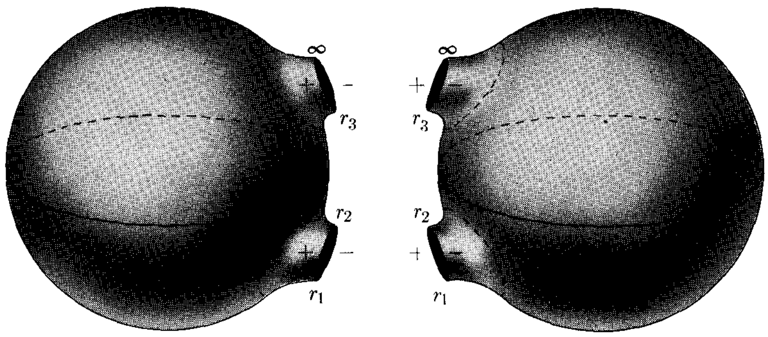

FIGURE 1–7.

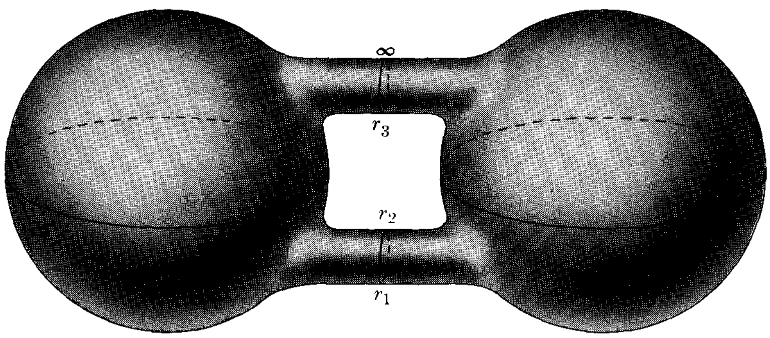

FIGURE 1–8.

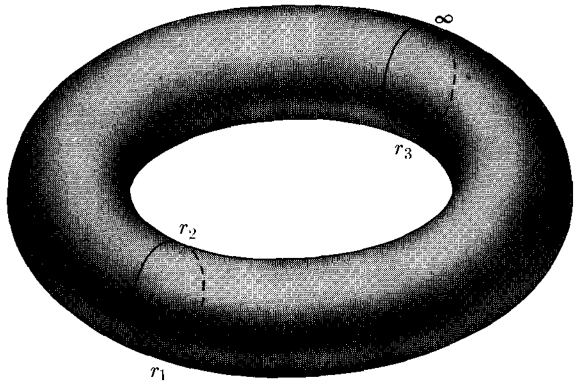

FIGURE 1-9.

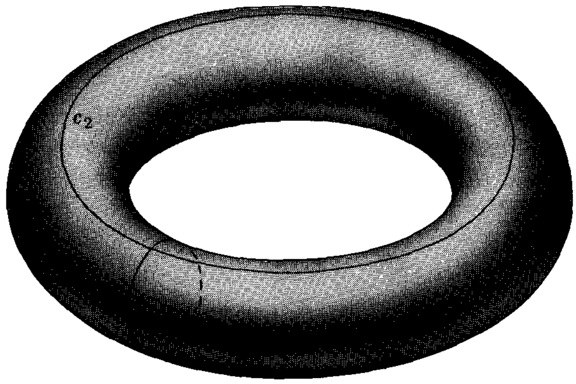

FIGURE 1-10.

indicated in Fig. 1-10 cannot be deformed continuously to a point on the surface of the torus. The curves marked C_1 and C_2 in Figs. 1-3 and 1-4 correspond to the meridian curves C_1 and latitude curves C_2 on the torus, respectively. In the two-sheeted Riemann surface of Fig. 1-4, the solid part of C_2 lies on one sheet and the dashed part on the other.

The existence on the surface of curves which, like C_1 and C_2, cannot be deformed to a point, affects the multiple-valuedness of the integrals of algebraic functions. Observe that around either C_1 or C_2, the function $w = \sqrt{a(z - r_1)(z - r_2)(z - r_3)}$ does not change its value.

In the cases studied previously, an integral

$$F(z) = \int_{z_0}^{z} R(z, w(z))\, dz,$$

where R is a rational function of z and w, had multiple-valuedness in the z-plane which arose because of the residues of R (logarithmic singularities of F) or because of the two-valuedness of $w(z)$. We shall soon see that

$$\int_{z_0}^{z} R(z, w(z))\, dz$$

can have a nonzero value around closed paths like C_1 and C_2 in Figs. 1–3 and 1–4 even though $w(z)$ remains single-valued on the curves and there are no residues of R enclosed by the curves. These integrals, with

$$w^2 = a(z - r_1)(z - r_2)(z - r_3),$$

are called *elliptic* integrals. The situation is similar when

$$w^2 = a(z - r_1)(z - r_2)(z - r_3)(z - r_4),$$

where r_1, r_2, r_3, r_4 are all distinct. In this case, cuts are made between r_1 and r_2 and between r_3 and r_4. These again can be opened and joined, as before, to give us a torus. Here also,

$$\int_{z_0}^{z} R(z, w)\, dz$$

is called an elliptic integral.

To complete the discussion of the special case $w^2 - p(z) = 0$, we take the function $w(z)$ defined by $w^2 = a(z - r_1)(z - r_2) \ldots (z - r_n)$, where the roots $r_1, r_2, \ldots, r_n$ are distinct. To each z correspond two values of w, so we get a two-sheeted Riemann surface with branch points at r_1, $r_2, \ldots, r_n$. As before, continuation of w along a path enclosing an odd number of the branch points leads to $-w$, while a path enclosing an even number of the branch points leads back to the original value of w. Thus, if we separate the branch points into pairs, say (r_1, r_2), $(r_3, r_4), \ldots$, and make cuts joining r_1 to r_2, r_3 to $r_4, \ldots$, we obtain two branches of $w(z)$, each single-valued in the cut plane. If n is odd, r_n is left over and we make a cut from r_n to ∞. This gives us $n/2$ cuts if n is even and $(n + 1)/2$ cuts if n is odd. If we connect two spheres, each cut between the branch points of w in pairs, as we did in the case $n = 3$ or 4, we obtain a surface such as that illustrated in Fig. 1–11. This surface consists of two spheres joined by $n/2$ tubes if n is even or $(n + 1)/2$ tubes if n is odd.

By momentarily fixing our attention on the two spheres and the one tube joining the cuts between r_1 and r_2 and closing the remaining cuts, we

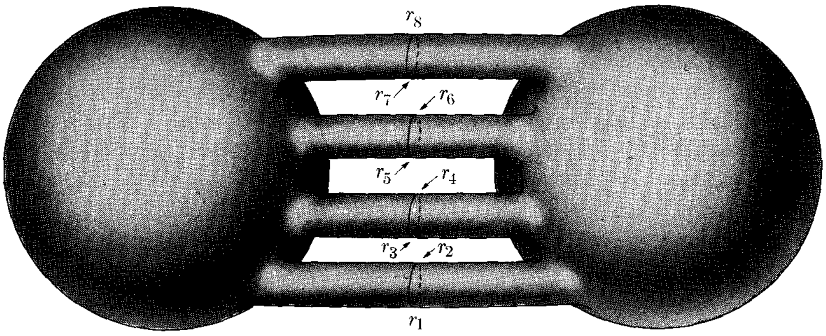

FIGURE 1-11.

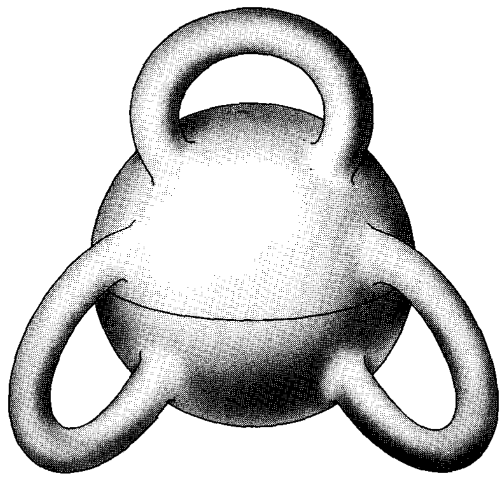

FIGURE 1-12.

obtain a surface which is topologically a sphere. Now we restore the remaining g tubes on this new sphere; here g is $(n/2) - 1$ if n is even and $(n+1)/2 - 1$ if n is odd. Each tube looks like a handle on the sphere; we get as the final topological model of the Riemann surface a sphere with g handles, as illustrated in Fig. 1-12. The number g is called the *genus* of the surface. Thus each algebraic function of the form $a_0(z)w^2 + a_1(z)w + a_2(z) = 0$, $a_0(z) \not\equiv 0$, has a Riemann surface which is topologically equivalent to a sphere with g handles. It can be shown that the Riemann surface for *any* algebraic function is topologically a sphere with g handles and that the algebraic function is a single-valued function of the points on this surface.