Ec. de Difusión en coordenadas esféricas

$$\frac{\partial \psi}{\partial t} = \alpha \nabla^2 \psi \qquad ; \quad \psi(r,\theta,\varphi,t)$$

$r \in [0,a]$

$\varphi \in [0,2\pi]$, $\theta \in [0,\pi]$

α = (conductividad térmica)

condiciones de frontera

$\psi(a,\theta,\varphi,t) = 0$ (Temp = 0 en la superficie)

ψ acotada en $r=0$, ψ acotada en $\theta = 0,\pi$, ψ periódica en φ

condición inicial

$$\psi\big|_{t=0} = f(r,\theta,\varphi)$$

solución separada.

$$\psi = T(t)\,R(r)\,P(\theta)\,Q(\varphi)$$

Sabemos que $P(\theta)Q(\varphi) = Y_\ell^m(\theta,\varphi)$. Además, si definimos el operador

$$-L^2 = \frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\operatorname{sen}^2\theta}\frac{\partial^2}{\partial\varphi^2}$$ (momento angular)

se tiene

$$L^2\, Y_\ell^m(\theta,\varphi) = \ell(\ell+1)\, Y_\ell^m(\theta,\varphi).$$

$L_z = -i\frac{\partial}{\partial\varphi}$ $\qquad L_z\, Y_\ell^m(\theta,\varphi) = m\, Y_\ell^m(\theta,\varphi)$

Tenemos entonces

$$\psi = T(t)\,R(r)\,Y_\ell^m(\theta,\varphi)$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{L^2}{r^2}\psi$$

$$\Rightarrow \nabla^2\psi = T\,Y_\ell^m\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R\right)$$

$$\frac{\partial\psi}{\partial t} = \alpha\nabla^2\psi \Rightarrow \frac{1}{\alpha}\frac{dT}{dt}\,R\,Y_\ell^m = T\,Y_\ell^m\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R\right)$$

dividiendo por $T R Y_\ell^m$

$$\frac{1}{\alpha}\frac{dT}{dt}\frac{1}{T} = \frac{1}{R}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R\right) = \lambda$$

De aquí se deducen dos EDO

I $\quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R = \lambda R \quad , \quad R(a)=0$

$R(r)$ acotada en $r=0$

II $\quad \frac{dT}{dt} = \alpha\lambda T$

I) $\quad AR = \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R = \lambda R \qquad$ problema S-L

$$a = 1 \;,\; p = \frac{2}{r} \;,\; h = \exp\left(\int p\,dr\right) = \exp(2\ln r) = r^2$$

$$w = \frac{h}{a} = r^2$$

se demuestra que A es autoadjunto en $[0,a]$ con las condiciones $R(a)=0$, $R(r)$ acotada en $r=0$

Ec. I - problema de Sturm-Liouville

$\lambda = 0$

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \ell(\ell+1)R = 0$$

$$R = A r^\ell + B r^{-(\ell+1)}$$

R finita en $r=0 \Rightarrow B=0$

$R(a)=0 \Rightarrow A=0$

$\lambda < 0$

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + (|\lambda| r^2 - \ell(\ell+1))R = 0$$

Ec. esférica de Bessel

$$R(r) = A\, j_\ell(\sqrt{|\lambda|}\, r) + B\, n_\ell(\sqrt{|\lambda|}\, r)$$

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x) \quad , \quad n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, N_{\ell+\frac{1}{2}}(x)$$

$j_\ell \underset{x\to 0}{\sim} x^\ell$ $\qquad$ n_ℓ diverge en $x=0$

R finita en $r=0 \Rightarrow B=0$

$R(a)=0 \Rightarrow \sqrt{|\lambda|}\, a = c_{\ell k} = \gamma_{\ell+\frac{1}{2},k}$, $j_\ell(c_{\ell k}) = 0$
$k=1,2,3,\dots$

algunos valores de $c_{\ell k}$

	$k=1$	$k=2$	$k=3$	$k=4$
$\ell=0$	3.142	6.283	9.425	12.566
$\ell=1$	4.493	7.725	10.904	14.066
$\ell=2$	5.763	9.095	12.323	15.515

notar $c_{0k} = k\pi$

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$\lambda > 0$

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - (\lambda r^2 + \ell(\ell+1))R = 0$$

ec. esférica modificada de Bessel

$$R(r) = A\, i_\ell(\sqrt{\lambda}\, r) + B\, k_\ell(\sqrt{\lambda}\, r)$$

R finita en $r=0 \Rightarrow B=0$.

$R(a) = 0 \Rightarrow A = 0$ ya que i_ℓ nunca se anula $(\lambda \neq 0)$

conclusión

$$R(r) = j_\ell\left(\frac{c_{\ell k}\, r}{a}\right)$$

conjunto completo en $[0,a]$, ortogonal con peso r^2

$$\lambda = -\frac{c_{\ell k}^2}{a^2}$$

Ec. II

$$\frac{dT}{dt} = -\alpha \frac{c_{\ell k}^2}{a^2} T$$

$$\Rightarrow T(t) = e^{-\alpha \frac{c_{\ell k}^2}{a^2} t}$$

SOLUCIÓN SEPARADA

$$\psi_{k\ell m} = e^{-\alpha \frac{c_{\ell k}^2}{a^2} t}\, j_\ell\left(\frac{c_{\ell k}\, r}{a}\right) Y_\ell^m(\theta,\varphi)$$

solución completa

$$\psi(r,\theta,\varphi,t) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{klm}\, e^{-\alpha \frac{c_{lk}^2}{a^2} t}\, j_l\left(\frac{c_{lk} r}{a}\right) Y_l^m(\theta,\varphi)$$

Los coeficientes A_{klm} se determinan a partir de

$$\psi\Big|_{t=0} = f(r,\theta,\varphi) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{klm}\, j_l\left(\frac{c_{lk} r}{a}\right) Y_l^m(\theta,\varphi)$$

$$A_{klm} = \frac{1}{\| j_l(\frac{c_{lk} r}{a})\|^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \operatorname{sen}\theta \int_0^a dr\, r^2\, Y_l^{*m}(\theta,\varphi)\, j_l\left(\frac{c_{lk} r}{a}\right) f(r,\theta,\varphi)$$

<u>nota</u>: norma de j_l

$$\| j_l\left(\frac{c_{lk} r}{a}\right)\|^2 = \int_0^a dr\, r^2\, j_l^2\left(\frac{c_{lk} r}{a}\right) = \frac{a^3}{2}\left[j_{l+1}(c_{lk})\right]^2$$

prueba

$$\int_0^a dr\, r^2\, j_l^2\left(\frac{c_{lk} r}{a}\right) = \frac{a^3}{c_{lk}^3}\int_0^{c_{lk}} dx\, x^2\, j_l^2(x)$$

$$= \frac{a^3}{c_{lk}^3}\int_0^{c_{lk}} dx\, x^2\left(\frac{\pi}{2x} J^2_{l+\frac{1}{2}}\right) = \frac{\pi a^3}{2 c_{lk}^3}\,\frac{c_{lk}^2}{2}\, J^2_{l+\frac{1}{2}+1}(c_{lk})$$

$$= \frac{a^3}{2}\left[j_{l+1}(c_{lk})\right]^2$$