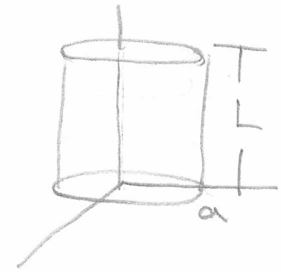
Ec. de Laplace en cilíndricas



2b) $\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = 0$

$\psi\big|_{\varphi=0} = \psi\big|_{\varphi=2\pi}$, $\frac{\partial\psi}{\partial\varphi}\Big|_{\varphi=0} = \frac{\partial\psi}{\partial\varphi}\Big|_{\varphi=2\pi}$ (I)

ψ acotada en $\rho=0$ (II)

$\psi\big|_{z=0}=0$, $\psi\big|_{z=L}=0$ (III) ; $\psi\big|_{\rho=a} = g(z,\varphi)$

$\psi_s(\rho,z,\varphi) = R(\rho)\,Z(z)\,Q(\varphi)$

condiciones de frontera homogéneas (II) $\Rightarrow$ $R(\rho)$ acotada en $\rho=0$

(I) $\Rightarrow$ $Q(0)=Q(2\pi)$; $\frac{dQ}{d\varphi}(0) = \frac{dQ}{d\varphi}(2\pi)$; (III) $\Rightarrow$ $Z(0)=0$, $Z(L)=0$

sustituyendo en la EDP y dividiendo por ψ_s

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\varphi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

La ec. se separa en 3 EDO's + condiciones de frontera

1. $\frac{d^2Z}{dz^2} = \lambda_1 Z$, $Z(0)=0$, $Z(L)=0$ problema S-L , $\lambda_1 = -\frac{n^2\pi^2}{L^2}$, $Z_n(z) = \operatorname{sen}\frac{n\pi z}{L}$ $\quad n=1,2,3..$

2. $\frac{d^2Q}{d\varphi^2} = \lambda_2 Q$, $Q(0)=Q(2\pi)$, $\frac{dQ}{d\varphi}(0) = \frac{dQ}{d\varphi}(2\pi)$, problema S-L . $\lambda_2 = -m^2$, $Q_m(\varphi) = e^{im\varphi}$ $\quad m=0,\pm1,\pm2,..$

3. $\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (\lambda_1\rho^2 + \lambda_2)R = 0$, $R(\rho)$ acotada en $\rho=0$

$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} - \left(\frac{n^2\pi^2}{L^2}\rho^2 + m^2\right)R = 0$ ec. modificada de Bessel.

$R(\rho) = A\,I_m\left(\frac{n\pi\rho}{L}\right) + B\,K_m\left(\frac{n\pi\rho}{L}\right)$, $R(\rho)$ acotada en $\rho=0$ $\Rightarrow$ $B=0$
$A\neq0$, se escoge $A=1$

soluciones separadas $\psi_{mn} = I_m\left(\frac{n\pi\rho}{L}\right)\operatorname{sen}\frac{n\pi z}{L}\,e^{im\varphi}$

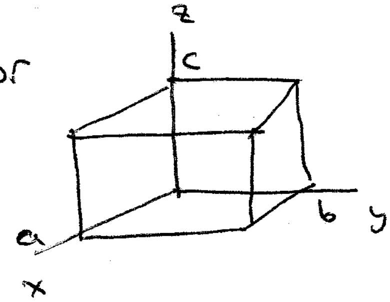
sol. completa $\psi(\rho,z,\varphi) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} C_{mn}\,\psi_{mn}(\rho,z,\varphi)$

$\psi\big|_{\rho=a} = g(z,\varphi) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \tilde{C}_{mn}\operatorname{sen}\frac{n\pi z}{L}\,e^{im\varphi}$, $\tilde{C}_{mn} = C_{mn}\,I_m\left(\frac{n\pi a}{L}\right)$

$$\tilde{C}_{mn} = \frac{\left(\operatorname{sen} \frac{n\pi z}{L} e^{im\varphi} , g(z,\varphi) \right)}{\left\| \operatorname{sen} \frac{n\pi z}{L} e^{im\varphi} \right\|^2}$$

$$C_{mn} = \frac{1}{\pi L I_m\left(\frac{n\pi a}{L}\right)} \int_0^{2\pi} d\varphi \int_0^{L} dz \, e^{-im\varphi} \operatorname{sen} \frac{n\pi z}{L} \, g(z,\varphi)$$

Ec. de Difusión en Cartesianas



Hallar la temperatura $\psi(x,y,z,t)$ en el interior de un cubo (ver figura) considerando que las caras del cubo se mantienen a temperatura cero e inicialmente

$$\psi(x,y,z,0) = f(x,y,z)$$

ψ satisface la ec. de difusion

$$\frac{1}{\alpha}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

, α : coeficiente de difusión (constante)

Las condiciones de frontera son

$$\psi\big|_{x=0} = 0 \,,\ \psi\big|_{x=a} = 0 \,,\ \psi\big|_{y=0} = 0 \,,\ \psi\big|_{y=b} = 0 \,,\ \psi\big|_{z=0} = 0 \,,\ \psi\big|_{z=c} = 0$$

separación de variables

$$\psi = X(x)\, Y(y)\, Z(z)\, T(t)$$

$$\frac{1}{\alpha}\dot{T} X Y Z = X'' Y Z T + X Y'' Z T + X Y Z'' T$$

$$\frac{1}{\alpha}\frac{\dot{T}}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}$$

$$\frac{1}{\alpha}\frac{\dot{T}}{T} - \frac{Y''}{Y} - \frac{Z''}{Z} = \frac{X''}{X} = \lambda_1$$

, etc.

La EDP se separa en 4 EDOs

1) $\frac{X''}{X} = \lambda_1$, $X(0) = X(a) = 0$

2) $\frac{Y''}{Y} = \lambda_2$, $Y(0) = Y(b) = 0$

3) $Z''/Z = \lambda_3$, $Z(0) = Z(c) = 0$

4) $\frac{\dot{T}}{T} = \alpha(\lambda_1 + \lambda_2 + \lambda_3)$

1, 2, 3 son problemas de S-L ya vistos. 3 se resuelve.

$$X_l(x) = \operatorname{sen} \frac{l\pi x}{a}, \quad \lambda_1 = -\frac{l^2\pi^2}{a^2}, \quad l = 1, 2, 3, \ldots$$

$$Y_m(y) = \operatorname{sen} \frac{m\pi y}{b}, \quad \lambda_2 = -\frac{m^2\pi^2}{b^2}, \quad m = 1, 2, 3, \ldots$$

$$Z_n(z) = \operatorname{sen} \frac{n\pi z}{c}, \quad \lambda_3 = -\frac{n^2\pi^2}{c^2}, \quad n = 1, 2, 3, \ldots$$

$$T_{lmn}(t) = e^{-\alpha k_{lmn} t}, \quad k_{lmn} = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{c^2}$$

La solución del problema completo es combinación lineal de las soluciones separadas

$$\psi(x, y, z, t) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{lmn} e^{-\alpha k_{lmn} t} \operatorname{sen} \frac{l\pi x}{a} \operatorname{sen} \frac{m\pi y}{b} \operatorname{sen} \frac{n\pi z}{c}$$

Los A_{lmn} se determinan a partir de la condición no-homogénea

$$\psi(x, y, z, 0) = f(x, y, z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{lmn} \operatorname{sen} \frac{l\pi x}{a} \operatorname{sen} \frac{m\pi y}{b} \operatorname{sen} \frac{n\pi z}{c}$$

Esta es una triple serie de Fourier seno

$$A_{lmn} = \frac{(f, X_l Y_m Z_n)}{\| X_l Y_m Z_n \|^2} = \frac{8}{abc} \int_0^a dx \int_0^b dy \int_0^c dz \; X_l(x)\, Y_m(y)\, Z_n(z)\, f(x, y, z)$$