PRUEBA DE $J_{-n}(z) = (-1)^n J_n(z)$

$\nu = -n-\epsilon$ (luego se toma $\epsilon \to 0$)

$$J_{-n-\epsilon}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1-n-\epsilon)} \left(\frac{z}{2}\right)^{2j-n-\epsilon}$$

$j-n = k$

$$J_{-n-\epsilon}(z) = \sum_{k=-n}^{\infty} \frac{(-1)^{k+n}}{(k+n)!\,\Gamma(1+k-\epsilon)} \left(\frac{z}{2}\right)^{2k+n-\epsilon}$$

$$= \underbrace{(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)!\,\Gamma(1+k-\epsilon)} \left(\frac{z}{2}\right)^{2k+n-\epsilon}}_{S_1} + \underbrace{(-1)^n \sum_{k=-n}^{-1} \frac{(-1)^k}{(k+n)!\,\Gamma(1+k-\epsilon)} \left(\frac{z}{2}\right)^{2k+n-\epsilon}}_{S_2}$$

consideremos la segunda suma

$$S_2 = \sum_{k=-n}^{-1} \frac{(-1)^k}{(k+n)!\,\Gamma(1+k-\epsilon)} \left(\frac{z}{2}\right)^{2k+n-\epsilon} = \sum_{r=1}^{n} \frac{(-1)^r}{(n-r)!\,\Gamma(1-r-\epsilon)} \left(\frac{z}{2}\right)^{n-2r-\epsilon}$$

ahora

$$\Gamma(1-r-\epsilon)\,\Gamma(r+\epsilon) = \frac{\pi}{\operatorname{sen}\pi(r+\epsilon)}$$

$$\operatorname{sen}(\pi r+\pi\epsilon) = \cancel{\operatorname{sen}\pi r}\cos\pi\epsilon + \cos\pi r\,\operatorname{sen}\pi\epsilon = (-1)^r \operatorname{sen}\pi\epsilon$$

$\operatorname{sen}\pi r \to 0$, r es entero

$$\Gamma(1-r-\epsilon) = \frac{(-1)^r\,\pi}{\operatorname{sen}\pi\epsilon\,\Gamma(r+\epsilon)}$$

$$S_2 = \frac{\operatorname{sen} \pi\epsilon}{\pi} \sum_{r=1}^{n} \frac{\Gamma(r+\epsilon)}{(n-r)!} \left(\frac{z}{2}\right)^{n-2r-\epsilon}$$

$$\lim_{\epsilon \to 0} S_2 = 0$$

También observamos que

$$\lim_{\epsilon \to 0} S_1 = (-1)^n J_n(z)$$

Forma Explícita de la Función de Neumann para $\nu = n$

$$N_n(x) = \lim_{\epsilon \to 0} \frac{J_{n+\epsilon}(x)\cos(n+\epsilon)\pi - J_{-(n+\epsilon)}(x)}{\operatorname{sen}(n+\epsilon)\pi}$$

$$= \lim_{\epsilon \to 0} \frac{J_{n+\epsilon}(x)\left[\cos n\pi \cos \epsilon\pi - \underbrace{\cancel{\operatorname{sen} n\pi}}_{0} \operatorname{sen}\epsilon\pi\right] - J_{-n-\epsilon}(x)}{\underbrace{\cancel{\operatorname{sen} n\pi}}_{0}\cos\epsilon\pi + \cos n\pi \operatorname{sen}\epsilon\pi}$$

$$= \lim_{\epsilon \to 0} \frac{J_{n+\epsilon}(x) - (-1)^n J_{-n-\epsilon}(x)}{\operatorname{sen}\epsilon\pi} \qquad \text{usando } \cos n\pi = (-1)^n$$

Usando L'Hôpital

$$N_n(x) = \frac{1}{\pi}\left[\frac{d}{d\epsilon} J_{n+\epsilon}(x) - (-1)^n \frac{d}{d\epsilon} J_{-n-\epsilon}(x)\right]\Bigg|_{\epsilon=0}$$

$$\frac{d}{d\epsilon} J_{n+\epsilon}(x) = \frac{d}{d\epsilon}\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1+n+\epsilon)}\left(\frac{x}{2}\right)^{2j+n+\epsilon}$$

$$= \ln\frac{x}{2}\, J_{n+\epsilon} - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1+n+\epsilon)}\psi(j+1+n+\epsilon)\left(\frac{x}{2}\right)^{2j+n+\epsilon}$$

donde se ha usado $\frac{d}{du}\Gamma(u) = \Gamma(u)\psi(u)$

$$\frac{d}{d\epsilon} J_{-n-\epsilon}(x) = -\ln\frac{x}{2}\, J_{-n-\epsilon} + \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1-n-\epsilon)}\psi(j+1-n-\epsilon)\left(\frac{x}{2}\right)^{2j-n-\epsilon}$$

Ahora

$$\sum_{j=0}^{\infty} \frac{(-1)^j \psi(j+1-n-\epsilon)}{j!\,\Gamma(j+1-n-\epsilon)} \left(\frac{x}{2}\right)^{2j-n-\epsilon} \underset{j-n=k}{=} \sum_{k=-n}^{\infty} \frac{(-1)^{k+n} \psi(k+1-\epsilon)}{\Gamma(k+n+1)\,\Gamma(k+1-\epsilon)} \left(\frac{x}{2}\right)^{2k+n-\epsilon}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n} \psi(k+1-\epsilon)}{\Gamma(k+n+1)\,\Gamma(k+1-\epsilon)} \left(\frac{x}{2}\right)^{2k+n-\epsilon} + \sum_{\substack{r=1 \\ k=-r}}^{n} \frac{(-1)^{n-r} \psi(1-r-\epsilon)}{\Gamma(1+n-r)\,\Gamma(1-r-\epsilon)} \left(\frac{x}{2}\right)^{-2r+n-\epsilon}$$

pero $\Gamma(1-r-\epsilon) = \dfrac{\pi(-1)^r}{\operatorname{sen}\pi\epsilon\,\Gamma(r)}$

$$\psi(r+\epsilon) - \psi(1-r-\epsilon) = -\frac{\pi\cos\pi\epsilon}{\operatorname{sen}\pi\epsilon}$$

$$\Rightarrow \lim_{\epsilon\to 0} \frac{\psi(1-r-\epsilon)}{\Gamma(1-r-\epsilon)} = (-1)^r\,\Gamma(r)$$

$$\Rightarrow \lim_{\epsilon\to 0} \frac{d}{d\epsilon} J_{-n-\epsilon} = -\ln\frac{x}{2}\,J_{-n}(x) + (-1)^n \sum_{j=0}^{\infty} \frac{(-1)^j \psi(j+1)}{j!\,(j+n)!} \left(\frac{x}{2}\right)^{2j+n}$$

$$+ (-1)^n \sum_{r=1}^{n} \frac{\Gamma(r)}{\Gamma(1+n-r)} \left(\frac{x}{2}\right)^{-2r+n} \qquad n-r=j$$

Usando $J_{-n}(x) = (-1)^n J_n(x)$ se obtiene finalmente

$$N_n(x) = \frac{2}{\pi} J_n(x) \ln\frac{x}{2} - \frac{1}{\pi} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{x}{2}\right)^{-n+2j}$$

ausente cuando $n=0$

$$-\frac{1}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+n)!} \left(\frac{x}{2}\right)^{2j+n} \left[\psi(j+n+1) + \psi(j+1)\right]$$

nota:

$$N_0(x) \xrightarrow[x\to 0]{} \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right)$$

usando $\psi(1) = -\gamma$

pero para $n \neq 0$

$$N_n(x) \xrightarrow[x\to 0]{} -\frac{1}{\pi}(n-1)!\left(\frac{x}{2}\right)^{-n}$$