On wave equations for the Majorana particle in (3+1) and (1+1) dimensions

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In general, the relativistic wave equation considered to mathematically describe the so-called Majorana particle is the Dirac equation with a real Lorentz scalar potential plus the so-called Majorana condition. Certainly, depending on the representation that one uses, the resulting differential equation changes. It could be a real or a complex system of coupled equations, or it could even be a single complex equation for a single component of the entire wave function. Any of these equations or systems of equations could be referred to as a Majorana equation or Majorana system of equations because it can be used to describe the Majorana particle. For example, in the Weyl representation, in (3+1) dimensions, we can have two non-equivalent explicitly covariant complex first-order equations; in contrast, in (1+1) dimensions, we have a complex system of coupled equations. In any case, whichever equation or system of equations is used, the wave function that describes the Majorana particle in (3+1) or (1+1) dimensions is determined by four or two real quantities. The aim of this paper is to study and discuss all these issues from an algebraic point of view, highlighting the similarities and differences that arise between these equations in the cases of (3+1) and (1+1) dimensions in the Dirac, Weyl, and Majorana representations. Additionally, to reinforce this task, we rederive and use results that come from a procedure already introduced by Case to obtain a two-component Majorana equation in (3+1) dimensions. Likewise, we introduce for the first time a somewhat analogous procedure in (1+1) dimensions and then use the results we obtain.

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I. INTRODUCTION

In general, the relativistic wave equation considered to mathematically describe a first quantized Majorana particle (an electrically neutral fermion in (3+1) dimensions that is its own antiparticle) is the Dirac equation with a real Lorentz scalar potential together with the so-called Majorana condition [1, 2]. The latter condition assumes that the Dirac wave function is equal to its respective charge-conjugate wave function, i.e., $\Psi = \Psi_C$; this is regardless of the representation of the gamma matrices that one chooses when writing the Dirac equation. This way of characterizing the Majorana particle can be implemented in (3+1) dimensions and also in (1+1) dimensions, although in the latter case one would be describing the one-dimensional Majorana particle.

As might be expected, depending on the representation that one uses when writing the Dirac equation and the Majorana condition, and without distinguishing between (3+1) and (1+1) dimensions, the differential equation that one obtains changes. It could be a real or a complex system of coupled equations or even a single complex equation for a single component of the entire wave function and whose solution, together with the relation that emerges from the Majorana condition, would allow one to build the entire wave function [3–6]. Certainly, any of these equations or systems of equations could be referred to as a Majorana equation or Majorana system of equations because any of them could be used to describe the Majorana particle.

Unexpectedly, the equation generally known in the literature as the Majorana equation is a relativistic wave equation similar to the free Dirac equation, $\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi - \frac{\mathrm{m}c}{\hbar}\hat{1}\Psi = 0$ ($\hat{1}$ is the identity matrix, which is a 4×4 matrix in (3+1) dimensions but a 2×2 matrix in (1+1) dimensions), but in addition to the Dirac wave function Ψ, the Majorana equation also includes the respective charge-conjugate wave function Ψ_C. The equation in question is usually written as $\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi - \frac{\mathrm{m}c}{\hbar}\hat{1}\Psi_C = 0$ [7], and by using typical properties associated with the charge conjugation operation, one obtains $\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi_C - \frac{\mathrm{m}c}{\hbar}\hat{1}\Psi = 0$; both of these equations imply that Ψ and Ψ_C satisfy the well-known Klein-Fock-Gordon equation. In writing the Majorana equation, it is important to remember that Ψ_C has the same transformation prop-

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erties as Ψ under proper Lorentz transformations; hence, this equation is Lorentz covariant. Likewise, the Majorana condition is Lorentz covariant [3]. The Majorana equation could describe hypothetical particles that have been called Majoranons [7]. Clearly, the Majorana equation together with the Majorana condition can also lead to equations for the Majorana particle [1]. It can be stated that the uncharged Majorana particle (for which $\Psi = \Psi_C$ is satisfied) would be the physical solution, and the charged Majoranon (for which the Majorana condition is not imposed) would be the unphysical solution of the Majorana equation [7]. We wish to point out in passing that the Majorana equation can also admit a Lorentz scalar potential.

In general, when characterizing a Majorana particle with the help of complex four-component wave functions (in (3+1) dimensions) or two-component wave functions (in (1+1) dimensions) these components are not all independent because the Majorana condition must be satisfied. Apropos of this, in the Majorana representation, the Majorana condition becomes the reality condition of the wave function, i.e., $\Psi = \Psi^{*}$; therefore, we can conclude that in (3+1) or (1+1) dimensions, the wave function that describes the Majorana particle has four or two independent real components, and these real components can be accommodated just in two or one independent complex components or component [3]. Then, to describe the Majorana particle in (3+1) or (1+1) dimensions, a four-component or two-component wave function is not absolutely necessary, i.e., a four-component or two-component scheme or formalism is not absolutely necessary; it can also be done with two-component or one-component wave functions, i.e., a two-component or one-component scheme or formalism in (3+1) or (1+1) dimensions is sufficient.

Returning to the issue of the equations for the Majorana particle that emerge from the Dirac equation and the Majorana condition when a representation is chosen, it is important to realize that in those cases where a complex first-order equation for the upper or lower single component of the entire wave function can be written (for example, in the Dirac representation), the respective lower or upper single component is automatically determined by the Majorana condition (depending on the space-time dimension, this single component can be a two-component or a one-component wave function). The entire wave function that describes the Majorana particle can be immediately constructed from these two components (the upper and the lower components). However, as explained above, the entire wave function is not absolutely needed to describe the Majorana particle; in fact, although the upper or lower component and its respective lower or upper component are not independent wave functions (i.e., they are not independent of each other), each of them satisfies its own equation, and either of these two can be considered as modeling the Majorana particle.

In (3+1) dimensions, there exists an equation for the upper component and another for the lower component that stand out above the rest (in this case, these components are two-component wave functions); these are the ones that arise when the Weyl representation is used. In fact, each of these equations can also be written in an explicitly Lorentz covariant form and can describe a specific type of Majorana particle. These equations have been named the two-component Majorana equations and tend to the usual Weyl equations when the mass of the particle and the scalar potential go to zero [8, 9]. Apropos of the latter result, in (1+1) dimensions and also in the Weyl representation, we have instead a complex system of coupled equations, i.e., in this case, we cannot write a first-order equation for any of the components of the wave function.

On the other hand, in (3+1) and (1+1) dimensions and in the Majorana representation, we also have a real system of coupled equations, and again, no first-order equation for any of the components of the wave function exists. Finally, the present contribution, beyond clarifying how the Majorana particle is described (in first quantization), also attempts to show the different forms of the equations that can arise when describing it, both in (3+1) and in (1+1) dimensions. We believe that a detailed discussion on these issues could be useful and quite pertinent.

The article is organized as follows. In section II, we present the most basic results that have to do with the relativistic wave equation commonly used to describe a Majorana particle, namely, the Dirac equation with a real Lorentz scalar potential. These results are presented for the cases of (3+1) and (1+1) dimensions.

In section III, we introduce the charge-conjugation matrix in each of the representations that we consider in the paper. We use only three representations, namely, Dirac (or the standard representation), Weyl (or the chiral or spinor representation) and Majorana. In practice, these are the most used; the first of these makes it very convenient to discuss the non-relativistic limit, the second makes immediately visible the relativistic invariance of the Dirac equation and is very useful for studying very fast particles, and the third could lead to real solutions for the Dirac equation with a real Lorentz scalar potential. Certainly, physics cannot depend on the choice of representation, although which representation is the best choice depends on the physics. In this section, the charge-conjugation matrices are obtained from a good formula that relates the matrices of charge conjugation in any two representations with the respective similarity matrix that changes the gamma matrices between these two representations. However, we specifically use the fact that in the Majorana representation, the charge-conjugation matrix is the identity matrix; thus, the charge-conjugation matrix in any representation is a function of the similarity matrix that takes us from that representation to the Majorana representation. Again, all these results are presented for the cases of (3+1) and (1+1) dimensions.

In section IV, we first present the condition that defines the Majorana particle, i.e., the Majorana condition. We then present the equations and systems of equations that come out of the Dirac equation with a real Lorentz scalar potential and the restriction imposed by the Majorana condition. Again, we consider the Dirac, Weyl, and Majorana representations, both in (3+1) and (1+1) dimensions. We also highlight here the similarities and differences that arise between these equations in a specific representation but in a different space-time dimension. In this regard, we note that, in the Weyl representation, there is a deeper and unexpected difference between these equations. Likewise, we highlight in this section the procedure that leads us in certain cases to write the entire wave function from the solution of a single equation and the Majorana condition (although the solution of this equation can be sufficient in the description of the Majorana particle). In this section, we also introduce, for the first time, various results related to the boundary conditions that can be imposed on the respective wave function that describes the one-dimensional Majorana particle in a box, in the Weyl representation.

To complete our study, in section V, we first rederive in detail an algebraic procedure introduced some time ago by Case to obtain, from the Dirac equation in (3+1) dimensions and the Majorana condition, one of the two two-component Majorana equations [8]. In fact, we also obtain the latter two equations after using the Weyl representation in our results, as expected. Moreover, we write these equations in distinct ways and compare these results with others commonly presented in the literature. Then, we also use the Dirac and Majorana representations in our results. In addition, we also introduce for the first time an algebraic procedure somewhat analogous to that of Case but this time in (1+1) dimensions. Then, we repeat the previous program by using the Weyl, Dirac, and Majorana representations in these new results. Throughout this section, we re-obtain the most important results presented in section IV. Finally, in section VI, we write our conclusions.

II. BASIC RESULTS

The equation for a Dirac single-particle in (3+1) dimensions, in a real-valued Lorentz scalar potential $V_{\mathrm{S}} = V_{\mathrm{S}}(x, y, z, t) = V_{\mathrm{S}}(\mathbf{r}, t)$,

$$\left[\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4 \right] \Psi = 0, \tag{1}$$

is satisfied by the (generally) complex Dirac wave function of four components Ψ. The matrix $\hat{1}_4$ is the 4-dimensional unit matrix. The matrices $\hat{\gamma}^{\mu} = (\hat{\gamma}^0, \hat{\gamma}^j) \equiv (\hat{\beta}, \hat{\beta}\hat{\alpha}_j)$, with $\mu = 0, j$ and $j = 1, 2, 3$, are the gamma matrices, and the matrices $\hat{\alpha}_j$ and $\hat{\beta}$ are the Dirac matrices. The latter are Hermitian and satisfy the (Clifford) relations $\{\hat{\alpha}_j, \hat{\beta}\} \equiv \hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j = \hat{0}_4$ ($\hat{0}_4$ is the 4-dimensional zero matrix), $\{\hat{\alpha}_j, \hat{\alpha}_k\} = 2\delta_{jk}\hat{1}_4$ and $\hat{\beta}^2 = \hat{1}_4$ (δ_{jk} is the Kronecker delta). Therefore, $\{\hat{\gamma}^{\mu}, \hat{\gamma}^{\nu}\} = 2g^{\mu\nu}\hat{1}_4$, where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor, and $(\hat{\gamma}^{\mu})^{\dagger} = \hat{\gamma}^0\hat{\gamma}^{\mu}\hat{\gamma}^0$ ($\dagger$ denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator, as usual). The latter two relations imply that the gamma matrices are unitary, but only $\hat{\gamma}^0$ is Hermitian, while $\hat{\gamma}^j$ is anti-Hermitian.

Multiplying Eq. (1) from the left by the operator $\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu} + \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4$ leads to the following second-order equation:

$$\left[\hat{1}_4\,\partial^{\mu}\partial_{\mu} + \frac{1}{\hbar c}(\partial_{\mu}V_{\mathrm{S}})\,\mathrm{i}\hat{\gamma}^{\mu} + \frac{(V_{\mathrm{S}} + \mathrm{m}c^2)^2}{\hbar^2 c^2}\hat{1}_4 \right] \Psi = 0. \tag{2}$$

Notice that the term containing $\hat{\gamma}^{\mu}$ is not generally a diagonal matrix, then the components of Ψ mix, In the free case ($V_{\mathrm{S}} = \mathrm{const}$), all the components satisfy the same equation, namely, the Klein-Fock-Gordon equation with mass $\mathrm{m}c^2 + \mathrm{const}$. Thus, the solutions of the Dirac equation with a Lorentz scalar potential, i.e., its components, must comply with a second-order equation.

The Dirac equation, written in its canonical form, is

$$\left(\mathrm{i}\hbar\hat{1}_4\,\frac{\partial}{\partial t} - \hat{\mathrm{H}} \right) \Psi = 0, \tag{3}$$

where the Hamiltonian operator $\hat{\mathrm{H}}$ is

$$\hat{\mathrm{H}} = -\mathrm{i}\hbar c \left(\hat{\alpha}_1\frac{\partial}{\partial x} + \hat{\alpha}_2\frac{\partial}{\partial y} + \hat{\alpha}_3\frac{\partial}{\partial z} \right) + (V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\beta}. \tag{4}$$

Eq. (3) is obtained from Eq. (1) by multiplying it by the matrix $\hbar c\hat{\gamma}^0 = \hbar c\hat{\beta}$ from the left, and using the relations $(\hat{\gamma}^0)^2 = \hat{1}_4$ and $\hat{\gamma}^0\hat{\gamma}^j = \hat{\alpha}_j$.

Likewise, Eq. (1) is also satisfied by the charge-conjugate wave function Ψ_C, but this yields

$$\hat{S}_C\,(-\hat{\gamma}^\mu)^*(\hat{S}_C)^{-1} = \hat{\gamma}^\mu\,, \quad \text{where} \quad \Psi_C \equiv \hat{S}_C\,\Psi^*, \tag{5}$$

and $\hat{S}_C$ is the charge-conjugation matrix (the asterisk * represents the complex conjugate) [10, 11]. This matrix is obviously determined up to a phase factor. As we said before, the matrices $\hat{\gamma}^\mu$ are unitary. More specifically, this is because $\{\hat{\gamma}^0, \hat{\gamma}^j\} = \hat{0}_4$ and $(\hat{\gamma}^0)^2 = -(\hat{\gamma}^j)^2 = \hat{1}_4$ and because $(\hat{\gamma}^0)^\dagger = \hat{\gamma}^0\hat{\gamma}^0\hat{\gamma}^0$ and $(\hat{\gamma}^j)^\dagger = \hat{\gamma}^0\hat{\gamma}^j\hat{\gamma}^0$. Likewise, the matrices $(-\hat{\gamma}^\mu)^*$ are also unitary. In effect, $g^{\mu\nu}$ is real; thus, we can write $(-\hat{\gamma}^\mu)^*(-\hat{\gamma}^\nu)^* + (-\hat{\gamma}^\nu)^*(-\hat{\gamma}^\mu)^* = 2g^{\mu\nu}\hat{1}_4$, and $((-\hat{\gamma}^\mu)^*)^\dagger = (-\hat{\gamma}^0)^*(-\hat{\gamma}^\mu)^*(-\hat{\gamma}^0)^*$; therefore, $((-\hat{\gamma}^0)^*)^\dagger = ((-\hat{\gamma}^0)^*)^{-1}$ and $((-\hat{\gamma}^j)^*)^\dagger = ((-\hat{\gamma}^j)^*)^{-1}$. Thus, because the matrices $\hat{\gamma}^\mu$ and $(-\hat{\gamma}^\mu)^*$ are linked via the relation on the left side of Eq. (5), the matrix $\hat{S}_C$ can be chosen to be unitary (for more details on this result, see, for example, Ref. [12], pag. 899). For example, in the Majorana representation, we have that $\Psi_C = \Psi^*$, i.e., $\hat{S}_C = \hat{1}_4$, and that $\hat{\gamma}^\mu = (-\hat{\gamma}^\mu)^* = \mathrm{i}\,\mathrm{Im}(\hat{\gamma}^\mu)$ (by virtue of Eq. (5)), i.e., all the entries of the gamma matrices are purely imaginary. Also, we have that $\mathrm{i}\hat{\gamma}^\mu = (\mathrm{i}\hat{\gamma}^\mu)^* = \mathrm{Re}(\mathrm{i}\hat{\gamma}^\mu)$, and consequently, the operator acting on Ψ in Eq. (1) is real. The latter condition implies only that Eq. (1) could have real-valued solutions. In the same way, Eq. (2) could also have real solutions.

All the equations and relations that we have written so far in (3+1) dimensions and that are dependent on Greek and Latin indices maintain their form in (1+1) dimensions. Certainly, these indices are now restricted to $\mu, \nu, \mathrm{etc} = 0, 1$, and $j, k, \mathrm{etc} = 1$. The Dirac wave function Ψ now has only two components and satisfies Eqs. (1), (2) and (3), with $\hat{1}_4 \to \hat{1}_2$ ($\hat{1}_2$ is the 2×2 identity matrix) also $V_{\mathrm{S}} = V_{\mathrm{S}}(x,t)$. The gamma matrices are just $\hat{\gamma}^0 \equiv \hat{\beta}$ and $\hat{\gamma}^1 \equiv \hat{\beta}\hat{\alpha}$, where the (Hermitian) Dirac matrices, $\hat{\alpha}$ and $\hat{\beta}$, satisfy the relations $\{\hat{\alpha}, \hat{\beta}\} = \hat{0}_2$ ($\hat{0}_2$ is the 2-dimensional zero matrix), $\hat{\alpha}^2 = \hat{1}_2$ and $\hat{\beta}^2 = \hat{1}_2$. Thus, $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2g^{\mu\nu}\hat{1}_2$, where $g^{\mu\nu} = \mathrm{diag}(1,-1)$, and $(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^0\hat{\gamma}^\mu\hat{\gamma}^0$. As before, the two gamma matrices are unitary, but $\hat{\gamma}^0$ is Hermitian, and $\hat{\gamma}^1$ is anti-Hermitian. Likewise, the Hamiltonian operator for the Dirac equation in Eq. (3) is simply given by

$$\hat{\mathrm{H}} = -\mathrm{i}\hbar c\,\hat{\alpha}\frac{\partial}{\partial x} + (V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\beta}. \tag{6}$$

III. CHARGE-CONJUGATION IN THE DIRAC, WEYL, AND MAJORANA REPRESENTATIONS

As is well known, in choosing a representation one is choosing a set of Dirac and gamma matrices that satisfies a Clifford relation (i.e., they form a Clifford algebra). As was demonstrated, for instance, in Ref. [6], if one has written the charge-conjugation matrix in a representation, let's say $\hat{S}_C$, then one can write it in any other representation, let's say $\hat{S}'_C$, via the following relation:

$$\hat{S}'_C = \hat{S}\,\hat{S}_C\,(\hat{S}^*)^{-1}, \tag{7}$$

where $\hat{S}$ is precisely the unitary similarity matrix that allows us to pass the unitary gamma matrices between these two representations, i.e., $\hat{\gamma}^{\mu\,\prime} = \hat{S}\,\hat{\gamma}^{\mu}\hat{S}^{-1}$. The result in Eq. (7) is simply due to the fact that the wave functions Ψ and Ψ_C are transformed under $\hat{S}$ as $\Psi' = \hat{S}\,\Psi$ and $\Psi'_C = \hat{S}\,\Psi_C$, but in each representation we also have that $\Psi_C \equiv \hat{S}_C\,\Psi^*$ and $\Psi'_C \equiv \hat{S}'_C\,(\Psi')^*$. Obviously, if we change the phase factor of the matrix $\hat{S}_C$, the matrix $\hat{S}'_C$ that is obtained from Eq. (7) changes in a factor that is also a phase. However, all the matrices involved in Eq. (7) are always determined up to an arbitrary phase factor. If we particularize the formula in Eq. (7) to the case in which $\hat{S}_C$ is written in an arbitrary representation and $\hat{S}'_C$ is written in the Majorana representation, i.e., $\hat{S}'_C = \hat{1}_4$, in (3+1) dimensions or $\hat{S}'_C = \hat{1}_2$, in (1+1) dimensions, one obtains the result

$$\hat{S}_C = \hat{S}^{\dagger}\hat{S}^*, \tag{8}$$

where $\hat{S}$ is the unitary matrix that takes us from that arbitrary representation to the Majorana representation. From Eq. (8), and because $\hat{S}_C$ is a unitary matrix, we obtain the result $(\hat{S}_C)^{-1} = (\hat{S}_C)^*$. The latter can also be obtained just by requiring that $(\Psi_C)_C = \Psi$.

The results pertinent to those representations usually identified as Dirac, Weyl, and Majorana in (3+1) dimensions are given in Table 1. The latter also shows the charge-conjugation matrix $\hat{S}_C$ in each representation derived from Eq. (8), the respective matrices $\hat{S}$ being the following:

$$\hat{S} = \frac{1}{\sqrt{2}}\left(\hat{\sigma}_x \otimes \hat{\sigma}_y \,+\, \hat{\sigma}_z \otimes \hat{1}_2\right), \tag{9}$$

which permits us to pass from the Dirac representation to the Majorana representation, and

$$\hat{S} = \frac{1}{2}\left(\hat{\sigma}_x \otimes \hat{\sigma}_y \,+\, \hat{\sigma}_z \otimes \hat{\sigma}_y \,+\, \hat{\sigma}_z \otimes \hat{1}_2 \,-\, \hat{\sigma}_x \otimes \hat{1}_2\right), \tag{10}$$

which permits us to pass from the Weyl representation to the Majorana representation. Obviously, the matrix $\hat{S}=\hat{1}_4=\hat{1}_2\otimes\hat{1}_2$ permits us to pass from the Majorana representation to the Majorana representation itself. Note that in (3+1) dimensions the charge-conjugation matrix in the Dirac representation is equal to the charge-conjugation matrix in the Weyl representation (up to a phase factor). For the sake of completeness, the matrix $\hat{S}$ that allows us to pass precisely from the Dirac representation to that of Weyl is also given here:

$$\hat{S}=\frac{1}{\sqrt{2}}\left(\hat{1}_2\otimes\hat{1}_2\,+\,\mathrm{i}\hat{\sigma}_y\otimes\hat{1}_2\right). \tag{11}$$

This matrix links the matrices $\hat{S}_C$ (in the Dirac representation) and $\hat{S}'_C$ (in the Weyl representation) also via Eq. (7).

In reading Tables 1, 1.1 and 1.2, the following definitions should be considered: $\hat{\boldsymbol{\alpha}}\equiv(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)$, $\hat{\boldsymbol{\gamma}}\equiv(\hat{\gamma}^1,\hat{\gamma}^2,\hat{\gamma}^3)$, and the usual Pauli matrices are $\hat{\boldsymbol{\sigma}}\equiv(\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z)$. $\otimes$ indicates the Kronecker product of matrices

$$\hat{A}\otimes\hat{B}\equiv\begin{bmatrix} a_{11}\hat{B} & \cdots & a_{1n}\hat{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\hat{B} & \cdots & a_{mn}\hat{B}\end{bmatrix}, \tag{12}$$

which satisfies the following properties: (i) $(\hat{A}\otimes\hat{B})(\hat{C}\otimes\hat{D})=(\hat{A}\hat{C}\otimes\hat{B}\hat{D})$, (ii) $(\hat{A}\otimes\hat{B})^{*}=\hat{A}^{*}\otimes\hat{B}^{*}$, and (iii) $(\hat{A}\otimes\hat{B})^{\dagger}=\hat{A}^{\dagger}\otimes\hat{B}^{\dagger}$ (for example, see Ref. [13]). Note that here we have $\hat{S}_C=-\hat{\gamma}^2$ in both the Dirac and Weyl representations. However, when considering these two representations, it is also common to write $\hat{S}_C=+\hat{\gamma}^2$, and in particle physics, it is more common to set $\hat{S}_C=+\mathrm{i}\hat{\gamma}^2$ and $\hat{S}_C=-\mathrm{i}\hat{\gamma}^2$.

In the same way, the results pertinent to those representations commonly considered as representations of Dirac, Weyl and Majorana in (1+1) dimensions are given in Table 2. The latter also shows the charge-conjugation matrix $\hat{S}_C$ in each representation calculated from Eq. (8). The respective matrices $\hat{S}$ are the following:

$$\hat{S}=\frac{1}{\sqrt{2}}\left(\hat{1}_2+\mathrm{i}\hat{\sigma}_x\right), \tag{13}$$

which permits us to pass from the Dirac representation to the Majorana representation, and

$$\hat{S}=\frac{1}{2}\left(\mathrm{i}\hat{1}_2+\hat{\sigma}_x+\hat{\sigma}_y+\hat{\sigma}_z\right), \tag{14}$$

which permits us to pass from the Weyl representation to the Majorana representation. Note that in (1+1) dimensions, the charge-conjugation matrix in the Dirac representation is not equal to the charge-conjugation matrix in the Weyl representation. For the sake of completeness, the matrix $\hat{S}$, which allows us to pass precisely from the Dirac representation to that of Weyl, is also given here:

$$\hat{S} = \frac{1}{\sqrt{2}} \left(\hat{\sigma}_x + \hat{\sigma}_z \right) . \tag{15}$$

This matrix links the matrices $\hat{S}_C$ (in the Dirac representation) and $\hat{S}'_C$ (in the Weyl representation) also via Eq. (7).

IV. EQUATIONS FOR THE MAJORANA SINGLE-PARTICLE I

The condition that defines a Majorana particle, called the Majorana condition, is given by

$$\Psi = \Psi_C = \hat{S}_C \, \Psi^* . \tag{16}$$

In general, the equation that describes this single particle is the Dirac equation (Eq. (1)) together with the latter condition. Apropos of this, it is important to note that the wave functions $\Psi \equiv [\,\text{top bottom}\,]^{\mathrm{T}}$ and $\Psi_C \equiv [\,\text{top}_C \ \text{bottom}_C\,]^{\mathrm{T}}$ (where the words "top" and "bottom" indicate the upper and lower components, respectively, of the respective wave function) are similarly transformed under proper Lorentz transformations ($^{\mathrm{T}}$ represents the transpose of a matrix). Thus, the upper components of these two wave functions, as well as the lower components, are similarly transformed. Obviously, this is true in any representation and has nothing to do with the Majorana condition. If in addition, the Majorana condition in Eq. (16) is verified, then the upper components of Ψ and Ψ_C, as well as their lower components, are equal. In passing, the Majorana condition is sometimes written as $\Psi = \omega \Psi_C$, where ω is an arbitrary unobservable phase factor, and it is still a Lorentz covariant condition [3], as expected. Below, we present the equations or systems of equations for the Majorana particle in the Dirac, Weyl and Majorana representations both in (3+1) and (1+1) dimensions. We make full use of Tables 1 and 2.

A. Dirac representation

In (3+1) dimensions. We write the four-component Dirac wave function (or Dirac spinor) Ψ in the form

$$\Psi \equiv \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \tag{17}$$

where the upper two-component wave function can be written as $\varphi \equiv [\,\varphi_1\ \varphi_2\,]^{\mathrm{T}}$ and the lower one as $\chi \equiv [\,\chi_1\ \chi_2\,]^{\mathrm{T}}$. In (3+1) dimensions, a two-component wave function such as Ψ is also called bispinor. The Dirac equation (Eq. (3)) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \hat{\mathrm{H}}\begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \begin{bmatrix} (V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2 & -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla \\ -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla & -(V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2 \end{bmatrix}\begin{bmatrix} \varphi \\ \chi \end{bmatrix}. \tag{18}$$

The Majorana condition (Eq. (16)) imposed upon the Dirac wave function imposes the following relation among the components of Ψ:

$$\chi = \hat{\sigma}_y\,\varphi^* \equiv \chi_C \quad (\Leftrightarrow\ \varphi = -\hat{\sigma}_y\,\chi^* \equiv \varphi_C). \tag{19}$$

Substituting the latter χ into Eq. (18), we are left with an equation for the two-component wave function φ, namely,

$$\mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi = -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\,(\hat{\sigma}_y\,\varphi^*) + (V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2\varphi. \tag{20}$$

Certainly, by making the latter replacement, two equations arise: one is Eq. (20), and the other is an equation that can also be obtained from Eq. (20) by making the following substitutions: $\varphi \to \hat{\sigma}_y\,\varphi^*$, $\hat{\sigma}_y\,\varphi^* \to \varphi$, and $V_{\mathrm{S}}+\mathrm{m}c^2 \to -(V_{\mathrm{S}}+\mathrm{m}c^2)$. Then, it can be algebraically shown that the latter equation and Eq. (20) are equivalent. Alternatively, if we substitute φ (from Eq. (19)) into Eq. (18), we obtain the following equation for the two-component wave function χ:

$$\mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\chi = -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\,(-\hat{\sigma}_y\,\chi^*) - (V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2\chi. \tag{21}$$

Again, by making the latter replacement, two equations arise: one is Eq. (21), and the other is an equation that can also be obtained from Eq. (21) by making the following replacements: $\chi \to -\hat{\sigma}_y\,\chi^*$, $-\hat{\sigma}_y\,\chi^* \to \chi$, and $-(V_{\mathrm{S}}+\mathrm{m}c^2) \to V_{\mathrm{S}}+\mathrm{m}c^2$. Again, it can be algebraically shown that the latter equation and Eq. (21) are absolutely equivalent. Clearly, if we assume that the wave function that describes the Majorana particle has four components, it is sufficient to solve at least one of the two last (decoupled) two-component equations, namely, Eqs. (20) and (21). This is because φ and χ are algebraically related by Eq. (19). Thus, Eq. (20) (or Eq. (21)) alone can be considered as a two-component equation that models the Majorana particle in (3+1) dimensions.

In Ref. [14], an equation analogous to Eq. (20), with $V_{\mathrm{S}} = 0$, was recently related to a nonlocal Schrödinger-type equation. Interestingly enough, the latter equation does not suffer from some of the problems that typically adversely affect relativistic single-particle equations. Incidentally, the authors in Ref. [14] used the same matrices corresponding to the Dirac representation shown in Table 1 but used a slightly different charge-conjugation matrix, namely, $\hat{S}_C = +\mathrm{i}\hat{\sigma}_y \otimes \hat{\sigma}_y$. Thus, the two-component equation used in that paper is precisely Eq. (20) with the following replacement: $\hat{\sigma}_y\,\varphi^* \to -\hat{\sigma}_y\,\varphi^*$. Likewise, in the same reference, two one-parameter families of confining boundary conditions were obtained for Majorana fermions restricted to a three-dimensional finite spatial domain.

In (1+1) dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (17), but in this case, φ and χ are simply functions of a single component. The Dirac equation (Eq. (3)) with the Hamiltonian in Eq. (6) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = \hat{\mathrm{H}}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = \begin{bmatrix}V_{\mathrm{S}}+\mathrm{m}c^2 & -\mathrm{i}\hbar c\frac{\partial}{\partial x}\\ -\mathrm{i}\hbar c\frac{\partial}{\partial x} & -(V_{\mathrm{S}}+\mathrm{m}c^2)\end{bmatrix}\begin{bmatrix}\varphi\\ \chi\end{bmatrix}. \tag{22}$$

The Majorana condition (Eq. (16)) imposed upon the Dirac wave function imposes the following relation between the two components of Ψ:

$$\chi = -\mathrm{i}\varphi^* \equiv \chi_C \quad (\Leftrightarrow\ \varphi = -\mathrm{i}\chi^* \equiv \varphi_C). \tag{23}$$

Substituting the latter χ into Eq. (22), we are left with an equation for the one-component wave function φ, namely,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi = -\mathrm{i}\hbar c\frac{\partial}{\partial x}(-\mathrm{i}\varphi^*) + (V_{\mathrm{S}}+\mathrm{m}c^2)\varphi \tag{24}$$

(the other equation that results after making the previous substitution in Eq. (22) is essentially the complex conjugate equation of Eq. (24)). Different from how it is in (3+1) dimensions, the equation for the lower component χ is simply equal to the equation for the upper component (Eq. (24)) but with the following replacement: $V_{\mathrm{S}} + \mathrm{m}c^2 \rightarrow -(V_{\mathrm{S}} + \mathrm{m}c^2)$. In any case, it is sufficient to solve at least one of these one-component equations because φ and χ are algebraically linked via Eq. (23). Thus, for example, it can be said that Eq. (24) (or the equation for χ) alone models the Majorana particle in (1+1) dimensions [6].

Again, in Ref. [14], an equation analogous to Eq. (24), with $V_{\mathrm{S}} = 0$, was related to a relativistic Schrödinger-type equation that has a consistent quantum mechanical single-particle interpretation (for example, it does not have negative energy states). In that reference, the authors used the same Dirac representation as we do, but the matrix $\hat{S}_C = +\mathrm{i}\hat{\sigma}_x$ was used instead; thus, the equation for the Majorana particle used therein is precisely Eq. (24) with the following replacement: $-\mathrm{i}\varphi^* \rightarrow +\mathrm{i}\varphi^*$. On the other hand, the only four boundary conditions that φ can support when the one-dimensional Majorana particle is inside an impenetrable box (we call them confining boundary conditions) were also encountered in Ref. [14]. Likewise, these conditions were found in Ref. [6], but it was shown in the latter reference that these are just the conditions that can arise mathematically from the general linear boundary condition used in the MIT bag model for a hadronic structure in (1+1) dimensions (certainly, the latter four boundary conditions are also subject to the Majorana condition). Specifically, for a box of size L with ends, for example, at $x = 0$ and $x = L$, the four confining boundary conditions can be written in the form $f(0,t) = g(L,t) = 0$, where f and g are the functions $\mathrm{Im}(\varphi)$ and $\mathrm{Re}(\varphi)$. Clearly, two of these boundary conditions are just the Dirichlet boundary condition imposed upon $\mathrm{Im}(\varphi)$ and $\mathrm{Re}(\varphi)$ at the ends of the box. The latter is a nice result because the entire two-component Dirac wave function does not support this type of boundary condition at the walls of the box [15]. In addition, two one-parameter families of non-confining boundary conditions, i.e., infinite non-confining boundary conditions (we call them non-confining because they do not cancel the probability current density at the ends of the box), were found in Ref. [6]. It is even possible (by taking some convenient limits) that these two families also include the four confining boundary conditions. Consequently, these two families actually make up the most general set of boundary conditions for the one-dimensional Majorana particle in a box; see Eq. (93) in Ref. [6]. In detail, we write below, for the first time, these two families of boundary conditions but in the Weyl representation.

Clearly, in the Dirac representation, the procedure for finding single equations for the Majorana particle is similar in (3+1) and (1+1) dimensions. However, this representation is not so commonly used to write the equation for the Majorana particle, neither in (3+1) nor in (1+1) dimensions; rather, Weyl's representation is used (at least in (3+1) dimensions).

B. Weyl representation

In (3+1) dimensions. We write the four-component Dirac wave function (or spinor) Ψ as follows:

$$\Psi \equiv \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{25}$$

where the upper (lower) two-component wave function can be written as $\varphi_1 \equiv [\,\xi_1\;\xi_2\,]^{\mathrm{T}}$ ($\varphi_2 \equiv [\,\xi_3\;\xi_4\,]^{\mathrm{T}}$). The Dirac equation (Eq. (3)) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \hat{\mathrm{H}}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla & -(V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2 \\ -(V_{\mathrm{S}}+\mathrm{m}c^2)\hat{1}_2 & +\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}. \tag{26}$$

The Majorana condition (Eq. (16)) imposed upon Ψ leads us to the following relation among its components:

$$\varphi_2 = \hat{\sigma}_y\,\varphi_1^* \equiv (\varphi_2)_C \quad (\Leftrightarrow\;\varphi_1 = -\hat{\sigma}_y\,\varphi_2^* \equiv (\varphi_1)_C\,). \tag{27}$$

Substituting the latter φ_2 into Eq. (26), we are left with an equation for the two-component wave function φ_1, namely,

$$\mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi_1 = -\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\varphi_1 - (V_{\mathrm{S}}+\mathrm{m}c^2)\,\hat{\sigma}_y\,\varphi_1^*. \tag{28}$$

Instead, if we substitute φ_1 (from Eq. (27)) into Eq. (26), we obtain the following equation for the two-component wave function φ_2:

$$\mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi_2 = +\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\varphi_2 + (V_{\mathrm{S}}+\mathrm{m}c^2)\,\hat{\sigma}_y\,\varphi_2^*. \tag{29}$$

Again, to obtain the wave function Ψ (Eq. (25)), it is sufficient to solve first Eq. (28) (Eq. (29)) to obtain φ_1 (φ_2) and then obtain φ_2 (φ_1) by using the Majorana condition (Eq. (27)). Note that the substitution that gave us Eq. (28) for φ_1 also generates another equation, namely, Eq. (29) for $\hat{\sigma}_y\,\varphi_1^*$ (these two equations are algebraically equivalent). Likewise, the substitution that gave us Eq. (29) for φ_2 also generates another equation, namely, Eq. (28) for $-\hat{\sigma}_y\,\varphi_2^*$ (again, both equations are equivalent). Thus, the wave function $\varphi_1 = \varphi_1(\mathbf{r},t)$ satisfies Eq. (28), but unexpectedly, $\mathrm{i}\hat{\sigma}_y\,\varphi_1^*(-\mathbf{r},t)$ also satisfies Eq. (28) (provided that the relation $V_{\mathrm{S}}(\mathbf{r},t)=V_{\mathrm{S}}(-\mathbf{r},t)$ is fulfilled). Similarly, the wave function $\varphi_2=\varphi_2(\mathbf{r},t)$ satisfies Eq. (29), but $-\mathrm{i}\hat{\sigma}_y\,\varphi_2^*(-\mathbf{r},t)$ also satisfies Eq. (29) (and again, the scalar potential must be an even function in $\mathbf{r}$).

Making $\mathrm{m}c^2=V_{\mathrm{S}}=0$ in Eq. (26), one obtains two (decoupled) equations, namely,

$$\mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi_1=-\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\varphi_1\,,\quad \mathrm{i}\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi_2=+\mathrm{i}\hbar c\,\hat{\boldsymbol{\sigma}}\cdot\nabla\varphi_2. \tag{30}$$

These are the well-known Weyl's equations. For instance, the first of these two-component equations can be assigned to the (right-handed, or right-helical) massless antineutrino, while the second one can be assigned to the (left-handed, or left-helical) massless neutrino (even though it is possible that only one of these two equations is sufficient for the description of a massless fermion, in which case one is led to the so-called Weyl theory [11, 16]). On the other hand, making $\mathrm{m}c^2=V_{\mathrm{S}}=0$ in Eqs. (28) and (29) one obtains two equations (in fact, the same equations given in Eq. (30)), but this time, φ_1 and φ_2 are related by the Majorana condition in Eq. (27). In fact, the four-component Majorana wave functions corresponding to the two-component wave functions φ_1 and φ_2 are given by

$$\Psi=\begin{bmatrix}\varphi_1\\ \hat{\sigma}_y\,\varphi_1^*\end{bmatrix}(=\Psi_C)\,,\quad\text{and}\quad \Psi=\begin{bmatrix}-\hat{\sigma}_y\,\varphi_2^*\\ \varphi_2\end{bmatrix}(=\Psi_C), \tag{31}$$

respectively. Meanwhile, the four-component Weyl wave functions corresponding to the two-component wave functions φ_1 and φ_2 are given by

$$\Psi=\begin{bmatrix}\varphi_1\\ 0\end{bmatrix},\quad\text{and}\quad \Psi=\begin{bmatrix}0\\ \varphi_2\end{bmatrix}, \tag{32}$$

respectively.

As can be seen in Table 1, we have considered the matrices $\hat{\boldsymbol{\alpha}}=+\hat{\sigma}_z\otimes\hat{\boldsymbol{\sigma}}$ and $\hat{\beta}=-\hat{\sigma}_x\otimes\hat{1}_2$ as the Weyl representation; however, in some books and articles, the matrices $\hat{\boldsymbol{\alpha}}'=+\hat{\sigma}_z\otimes\hat{\boldsymbol{\sigma}}$ and $\hat{\beta}'=+\hat{\sigma}_x\otimes\hat{1}_2$ are also used as a Weyl representation (for example, in Refs. [9, 17]). Even the matrices $\hat{\boldsymbol{\alpha}}'=-\hat{\sigma}_z\otimes\hat{\boldsymbol{\sigma}}$ and $\hat{\beta}'=+\hat{\sigma}_x\otimes\hat{1}_2$ have been considered as a Weyl representation in other publications (for example, in Ref. [3]). As usual, the former and the latter two representations are related by $\hat{\boldsymbol{\alpha}}'=\hat{S}\,\hat{\boldsymbol{\alpha}}\,\hat{S}^\dagger$ and $\hat{\beta}'=\hat{S}\,\hat{\beta}\,\hat{S}^\dagger$ as well as $[\,\varphi_1'\;\varphi_2'\,]^{\mathrm{T}}=\hat{S}\,[\,\varphi_1\;\varphi_2\,]^{\mathrm{T}}$, but we must use $\hat{S}=\hat{\sigma}_z\otimes\hat{1}_2=\hat{S}^\dagger$ to relate the first and the second pair of Dirac matrices and $\hat{S}=\hat{\sigma}_y\otimes\hat{1}_2=\hat{S}^\dagger$ to relate the first and the third pair.

In (3+1) dimensions, the Weyl representation is definitely the most used. As we will show in section V, the two-component Eqs. (28) and (29) can also be written explicitly in covariant form, and each of them can describe a Majorana particle.

In (1+1) dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (25), but in this case, φ_1 and φ_2 are wave functions of a single component. The Dirac equation (Eq. (3)) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\hat{\mathrm{H}}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=\begin{bmatrix}-\mathrm{i}\hbar c\frac{\partial}{\partial x} & V_{\mathrm{S}}+\mathrm{m}c^2\\ V_{\mathrm{S}}+\mathrm{m}c^2 & +\mathrm{i}\hbar c\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}. \tag{33}$$

The Majorana condition (Eq. (16)) imposed upon Ψ gives us the following relations:

$$\varphi_1=-\mathrm{i}\varphi_1^*\equiv(\varphi_1)_C \text{ and } \varphi_2=+\mathrm{i}\varphi_2^*\equiv(\varphi_2)_C. \tag{34}$$

Obviously, these relations do not allow us to write a one-component first-order equation for the Majorana particle (and from Eq. (2), neither can a standard one-component second-order equation be written). That is, unlike what happens in (3+1) dimensions, the equation that describes the Majorana particle in (1+1) dimensions is a complex system of coupled equations, i.e., Eq. (33) with the restriction given in Eq. (34).

In this representation, we can also write the most general set of boundary conditions for the one-dimensional Majorana particle inside a box with ends at $x=0$ and $x=L$. This set consists of two one-parameter families of boundary conditions. In fact, using the results given in Eqs. (67) and (68) of Ref. [6] (written in the Majorana representation) and the fact that the two-component wave functions in the Weyl and Majorana representations verify the relation $[\,\phi_1\;\phi_2\,]^{\mathrm{T}}=\hat{S}\,[\,\varphi_1\;\varphi_2\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (14), we obtain, respectively (we exclude the variable t in the boundary conditions hereinafter),

$$\begin{bmatrix}\varphi_1(L)\\ \varphi_2(L)\end{bmatrix}=\frac{1}{m_2}\begin{bmatrix}-1 & -\mathrm{i}m_0\\ -\mathrm{i}m_0 & +1\end{bmatrix}\begin{bmatrix}\varphi_1(0)\\ \varphi_2(0)\end{bmatrix}, \tag{35}$$

where $(m_0)^2+(m_2)^2=1$, and

$$\begin{bmatrix}\varphi_1(L)\\ \varphi_2(L)\end{bmatrix}=\frac{1}{m_1}\begin{bmatrix}+1 & -\mathrm{i}m_3\\ +\mathrm{i}m_3 & +1\end{bmatrix}\begin{bmatrix}\varphi_1(0)\\ \varphi_2(0)\end{bmatrix}, \tag{36}$$

where $(m_1)^2+(m_3)^2=1$. Note that the 2×2 matrix in (35) is equal to its own inverse and that the inverse matrix of the 2×2 matrix in (36) is obtained from the latter by making the substitution $m_3 \to -m_3$. We obtain two boundary conditions for an impenetrable box (i.e., two confining boundary conditions) from Eq. (35) and its inverse by making $m_2 \to 0$, namely,

$$\varphi_1(L) = -\mathrm{i}\varphi_2(L)\,, \quad \varphi_1(0) = -\mathrm{i}\varphi_2(0), \tag{37}$$

with $m_0 = 1$, and

$$\varphi_1(L) = +\mathrm{i}\varphi_2(L)\,, \quad \varphi_1(0) = +\mathrm{i}\varphi_2(0), \tag{38}$$

with $m_0 = -1$. Likewise, we obtain two other confining boundary conditions from Eq. (36) and its inverse by making $m_1 \to 0$, namely,

$$\varphi_1(L) = -\mathrm{i}\varphi_2(L)\,, \quad \varphi_1(0) = +\mathrm{i}\varphi_2(0), \tag{39}$$

with $m_3 = 1$, and

$$\varphi_1(L) = +\mathrm{i}\varphi_2(L)\,, \quad \varphi_1(0) = -\mathrm{i}\varphi_2(0), \tag{40}$$

with $m_3 = -1$. Note that the wave function $[\,\varphi_1\;\varphi_2\,]^{\mathrm{T}}$ can satisfy any of the boundary conditions included in Eqs. (35) and (36), but then the wave function $[\,-\mathrm{i}\varphi_1^*\;+\mathrm{i}\varphi_2^*\,]^{\mathrm{T}}$ also automatically satisfies this boundary condition. This is due to the Majorana condition. Because in this case the Majorana condition is a pair of independent relations, the boundary conditions are presented in terms of the two components of the wave function.

C. Majorana representation

In (3+1) dimensions. The four-component Dirac wave function (or spinor) Ψ can be written as

$$\Psi \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{41}$$

where the upper (lower) two-component wave function could be written as $\phi_1 \equiv [\,\zeta_1\;\zeta_2\,]^{\mathrm{T}}$ ($\phi_2 \equiv [\,\zeta_3\;\zeta_4\,]^{\mathrm{T}}$). The Dirac equation (Eq. (3)) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \hat{\mathrm{H}}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\mathrm{i}\hbar c\hat{1}_2\frac{\partial}{\partial y} & +\mathrm{i}\hbar c\left(\hat{\sigma}_x\frac{\partial}{\partial x} + \hat{\sigma}_z\frac{\partial}{\partial z}\right) + (V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\sigma}_y \\ +\mathrm{i}\hbar c\left(\hat{\sigma}_x\frac{\partial}{\partial x} + \hat{\sigma}_z\frac{\partial}{\partial z}\right) + (V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\sigma}_y & +\mathrm{i}\hbar c\hat{1}_2\frac{\partial}{\partial y} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{42}$$

Clearly, Eq. (42) is a real system of two coupled equations for the two-component wave functions ϕ_1 and ϕ_2. Thus, one can obtain real-valued solutions for this equation, but complex-valued solutions can also be obtained (although these do not describe a Majorana particle) [18]. The Majorana condition (Eq. (16)) imposed upon Ψ leads us to the following relation:

$$\Psi = \Psi^* \,(\Leftrightarrow\; \phi_1 = \phi_1^* \equiv (\phi_1)_C \text{ and } \phi_2 = \phi_2^* \equiv (\phi_2)_C\,). \tag{43}$$

That is, the Majorana condition imposed on the Dirac wave function in the Majorana representation is what implies that this wave function must be real.

In passing, we note that the matrices originally chosen by Majorana in his 1937 article were the following [1]: $\hat{\alpha}'_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{\alpha}'_2 = \hat{\sigma}_z \otimes \hat{1}_2$, $\hat{\alpha}'_3 = \hat{\sigma}_x \otimes \hat{\sigma}_z$ and $\hat{\beta}' = -\,\hat{\sigma}_x \otimes \hat{\sigma}_y$. Of these four matrices, only $\hat{\alpha}'_2$ coincides with our matrix $\hat{\alpha}_2$. The other three matrices differ from ours by a minus sign. Incidentally, these two representations are related by $\hat{\boldsymbol{\alpha}}' = \hat{S}\,\hat{\boldsymbol{\alpha}}\,\hat{S}^\dagger$, $\hat{\beta}' = \hat{S}\,\hat{\beta}\,\hat{S}^\dagger$, and $[\,\phi'_1\;\phi'_2\,]^{\mathrm{T}} = \hat{S}\,[\,\phi_1\;\phi_2\,]^{\mathrm{T}}$, where $\hat{S} = \hat{\sigma}_z \otimes \hat{1}_2 = \hat{S}^\dagger$.

In (1+1) dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (41), but in this case, ϕ_1 and ϕ_2 are simply functions of a single component. The Dirac equation (Eq. (3)) has the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \hat{\mathrm{H}}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{i}\hbar c\frac{\partial}{\partial x} - \mathrm{i}(V_{\mathrm{S}} + \mathrm{m}c^2) \\ -\mathrm{i}\hbar c\frac{\partial}{\partial x} + \mathrm{i}(V_{\mathrm{S}} + \mathrm{m}c^2) & 0 \end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}. \tag{44}$$

Again, the Dirac equation in this representation is a real system of two coupled equations for the wave functions ϕ_1 and ϕ_2. However, it is precisely the Majorana condition (Eq. (16)) imposed upon Ψ that leads us to the real condition of the wave function:

$$\Psi = \Psi^* \,(\Leftrightarrow\; \phi_1 = \phi_1^* \equiv (\phi_1)_C \text{ and } \phi_2 = \phi_2^* \equiv (\phi_2)_C\,). \tag{45}$$

Recently, distinct real-valued general solutions of the time-dependent Dirac equation in Eq. (44) (i.e., subject to the constraint in Eq. (45)), for distinct scalar potentials and borders, were constructed [18]. Certainly, all these solutions describe a one-dimensional Majorana particle in its respective physical situation.

The most general set of boundary conditions for the one-dimensional Majorana particle inside a box in the Majorana representation was written in detail in Ref. [6]. This set consists of two real one-parameter families of boundary conditions (see Eqs. (67) and (68) of that reference). The Majorana condition in the Majorana representation leads very easily to the Majorana condition in any other representation. In fact, we know that wave functions in the Dirac and Majorana representations are linked through the relation $[\,\phi_1\;\phi_2\,]^{\mathrm{T}} = \hat{S}\,[\,\varphi\;\chi\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (13); in addition, wave functions in the Weyl and Majorana representations are linked through the relation $[\,\phi_1\;\phi_2\,]^{\mathrm{T}} = \hat{S}\,[\,\varphi_1\;\varphi_2\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (14). Thus, by imposing the Majorana condition (Eq. (45)) on the latter two relations, we obtain the Majorana condition in the Dirac and Weyl representations, i.e., Eqs. (23) and (34), respectively. Certainly, the latter general discussion is also valid in (3+1) dimensions.

V. EQUATIONS FOR THE MAJORANA SINGLE-PARTICLE II

In (3+1) dimensions. Let us define, as Case did [8], the following wave functions and matrices:

$$\Psi_{\pm} \equiv \frac{1}{2}\left(\hat{1}_4 \pm \hat{\gamma}^5\right)\Psi \tag{46}$$

and

$$\hat{\gamma}^{\mu}_{\pm} \equiv \frac{1}{2}\left(\hat{1}_4 \pm \hat{\gamma}^5\right)\hat{\gamma}^{\mu}, \tag{47}$$

where the matrix $\hat{\gamma}^5 \equiv \mathrm{i}\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = -\mathrm{i}\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3$ is Hermitian and satisfies the relations $(\hat{\gamma}^5)^2 = \hat{1}_4$, and $\{\hat{\gamma}^5, \hat{\gamma}^{\mu}\} = \hat{0}_4$. In addition, $\hat{\gamma}^5$ satisfies the relation $\hat{S}_C\,(-\hat{\gamma}^5)^*(\hat{S}_C)^{-1} = \hat{\gamma}^5$ (i.e., $\hat{\gamma}^5$, just as $\hat{\gamma}^{\mu}$, satisfies Eq. (5)), and

$$\left[\frac{1}{2}\left(\hat{1}_4 \pm \hat{\gamma}^5\right)\right]^2 = \frac{1}{2}\left(\hat{1}_4 \pm \hat{\gamma}^5\right)\,, \quad \text{and} \quad \frac{1}{2}\left(\hat{1}_4 \pm \hat{\gamma}^5\right)\frac{1}{2}\left(\hat{1}_4 \mp \hat{\gamma}^5\right) = \hat{0}_4. \tag{48}$$

Note that the charge conjugate of the wave functions in (46) verify $(\Psi_{\pm})_C = (\Psi_C)_{\mp}$. The matrix $\hat{\gamma}^5$ is called the chirality matrix and its eigenstates are precisely Ψ_+ (the right-chiral state), with eigenvalue $+1$, and Ψ_- (the left-chiral state), with eigenvalue -1 [3] (the latter two results can easily be demonstrated by multiplying Eq. (46) by $\hat{\gamma}^5$ from the left). However, also note that $(\Psi_+)_C$ is the eigenstate of $\hat{\gamma}^5$ with eigenvalue -1 (i.e., it is a left-chiral state), and $(\Psi_-)_C$ is the eigenstate of $\hat{\gamma}^5$ with eigenvalue $+1$ (i.e., it is a right-chiral state). The matrices $\hat{\gamma}^5$ and the wave functions $\Psi_{\pm}$ in the three representations that we use in this article are shown in Tables 1 and 3, respectively.

First, by multiplying Eq. (1) by $\frac{1}{2}(\hat{1}_4 + \hat{\gamma}^5)$ from the left, we obtain the equation

$$\mathrm{i}\hat{\gamma}^{\mu}_{+}\partial_{\mu}\Psi_{-} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4\Psi_{+} = 0, \tag{49}$$

and similarly, by multiplying Eq. (1) by $\frac{1}{2}(\hat{1}_4 - \hat{\gamma}^5)$, we obtain the equation

$$\mathrm{i}\hat{\gamma}^{\mu}_{-}\partial_{\mu}\Psi_{+} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4\Psi_{-} = 0. \tag{50}$$

Because $\Psi = \Psi_{+} + \Psi_{-}$ and $\hat{\gamma}^{\mu} = \hat{\gamma}^{\mu}_{+} + \hat{\gamma}^{\mu}_{-}$, we have that Eqs. (49) and (50) are completely equivalent to the Dirac equation (1). Likewise, because Eq. (1) is also satisfied by the charge-conjugate wave function, we also have two equations that are equivalent to the Dirac equation for Ψ_C. In effect, multiplying the latter by $\frac{1}{2}(\hat{1}_4 + \hat{\gamma}^5)$ and $\frac{1}{2}(\hat{1}_4 - \hat{\gamma}^5)$, we obtain

$$\mathrm{i}\hat{\gamma}^{\mu}_{+}\partial_{\mu}(\Psi_{+})_C - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4(\Psi_{-})_C = 0\,, \quad \text{and} \quad \mathrm{i}\hat{\gamma}^{\mu}_{-}\partial_{\mu}(\Psi_{-})_C - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4(\Psi_{+})_C = 0, \tag{51}$$

respectively (remember that $(\Psi_{\pm})_C = \hat{S}_C\Psi^{*}_{\pm}$). Note that because $(\Psi_{+})_C = (\Psi_C)_{-}$ and $(\Psi_{-})_C = (\Psi_C)_{+}$, the wave functions Ψ_{+} and Ψ_{-} as well as $(\Psi_C)_{+}$ and $(\Psi_C)_{-}$, satisfy the same system of coupled equations, namely, Eqs. (49) and (50) (or the system in Eq. (51)), as expected.

In the case where $\mathrm{m}c^2 = V_{\mathrm{S}} = 0$, Eqs. (49) and (50) are decoupled, and we have $\mathrm{i}\hat{\gamma}^{\mu}_{+}\partial_{\mu}\Psi_{-} = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi_{-} = 0$) and $\mathrm{i}\hat{\gamma}^{\mu}_{-}\partial_{\mu}\Psi_{+} = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi_{+} = 0$). In the Weyl representation, the latter two four-component equations give us the usual Weyl equations (Eq. (30)). In the same way, if we make $\mathrm{m}c^2 = V_{\mathrm{S}} = 0$ in the system in Eq. (51), then we obtain $\mathrm{i}\hat{\gamma}^{\mu}_{+}\partial_{\mu}(\Psi_{+})_C = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}(\Psi_{+})_C = 0$) and $\mathrm{i}\hat{\gamma}^{\mu}_{-}\partial_{\mu}(\Psi_{-})_C = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}(\Psi_{-})_C = 0$). Certainly, in the Weyl representation, the latter two equations also give us the usual Weyl equations (Eq. (30)).

The Majorana condition in Eq. (16) takes the form

$$\Psi_{-} = (\Psi_{+})_C \;(\Leftrightarrow\; \Psi_{+} = (\Psi_{-})_C\,) \tag{52}$$

(remember that $(\hat{S}_C)^{-1} = (\hat{S}_C)^{*}$), i.e., $\Psi_{-} = (\Psi_C)_{-}$ ($\Leftrightarrow$ $\Psi_{+} = (\Psi_C)_{+}$). Substituting the latter Ψ_{-} into Eq. (50), we obtain an equation for the four-component wave function Ψ_{+}, namely,

$$\mathrm{i}\hat{\Gamma}^{\mu}\partial_{\mu}\Psi_{+} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4\Psi^{*}_{+} = 0, \tag{53}$$

where

$$\hat{\Gamma}^{\mu} \equiv (\hat{S}_C)^{*}\,\hat{\gamma}^{\mu}_{-}\,, \quad \text{with} \quad (\hat{\Gamma}^{\mu})^{*}\,\hat{\Gamma}^{\nu} + (\hat{\Gamma}^{\nu})^{*}\,\hat{\Gamma}^{\mu} = -2g^{\mu\nu}\frac{1}{2}\left(\hat{1}_4 + \hat{\gamma}^5\right) \tag{54}$$

(the equation for Ψ_{+} that results after making the latter substitution but into Eq. (49) is absolutely equivalent to Eq. (53)).

Alternatively, substituting the wave function Ψ_+ from Eq. (52) into Eq. (49), we obtain an equation for the four-component wave function Ψ_-, namely,

$$\mathrm{i}\hat{\Lambda}^{\mu}\partial_{\mu}\Psi_- - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4\Psi_-^* = 0, \tag{55}$$

where

$$\hat{\Lambda}^{\mu} \equiv (\hat{S}_C)^*\,\hat{\gamma}_+^{\mu}\,, \quad \text{with} \quad (\hat{\Lambda}^{\mu})^*\,\hat{\Lambda}^{\nu} + (\hat{\Lambda}^{\nu})^*\,\hat{\Lambda}^{\mu} = -2g^{\mu\nu}\frac{1}{2}\left(\hat{1}_4 - \hat{\gamma}^5\right) \tag{56}$$

(again, the equation for Ψ_- that results after making the latter substitution but into Eq. (50) is absolutely equivalent to Eq. (55)). Naturally, by imposing the Majorana condition (Eq. (52)) upon the equations in (51), we again obtain Eqs. (53) and (55).

On the other hand, making $\mathrm{m}c^2 = V_{\mathrm{S}} = 0$ in Eq. (53) leads us to the relation $\mathrm{i}\hat{\gamma}_-^{\mu}\partial_{\mu}\Psi_+ = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi_+ = 0$), and as can be seen in Eq. (51), we also have $\mathrm{i}\hat{\gamma}_-^{\mu}\partial_{\mu}(\Psi_-)_C = 0$ ($\Rightarrow$ $\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}(\Psi_-)_C = 0$), but also in this case, we have $\Psi_+ = (\Psi_-)_C$ (this is due to the Majorana condition). Similarly, making $\mathrm{m}c^2 = V_{\mathrm{S}} = 0$ in Eq. (55) leads us to the relation $\mathrm{i}\hat{\gamma}_+^{\mu}\partial_{\mu}\Psi_- = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi_- = 0$), but from Eq. (51) we also have $\mathrm{i}\hat{\gamma}_+^{\mu}\partial_{\mu}(\Psi_+)_C = 0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}(\Psi_+)_C = 0$), where $\Psi_- = (\Psi_+)_C$ (this is also due to the Majorana condition).

To obtain the four-component wave function that describes the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_-$, it is sufficient to solve the equation for Ψ_+ (Eq. (53)), and then, from this solution, and using the Majorana condition in (52), one obtains Ψ_-. Alternatively, one could also solve the equation for Ψ_- (Eq. (55)), and then, from this solution, and using the Majorana condition in (52), one obtains Ψ_+. Note that, in the former case, $\Psi = \Psi_+ + (\Psi_+)_C$, and therefore, $\Psi = \Psi_C$ (remember that $((\Psi_+)_C)_C = \Psi_+$); similarly, in the latter case, $\Psi = (\Psi_-)_C + \Psi_-$, and therefore, $\Psi = \Psi_C$ (remember that $((\Psi_-)_C)_C = \Psi_-$), as expected. Clearly, the four-component wave function Ψ depends only on the solution of Eq. (53) (or of Eq. (55)); thus, we can consider that Eq. (53) (or Eq. (55)) alone models the Majorana particle in (3+1) dimensions and in a form independent of the choice of representation.

Certainly, the above-mentioned procedure to obtain Ψ is general, but in each representation, it has its own particularity. In relation to this, we can now obtain different results. In the rest of this subsection, we make full use of Tables 3, 4 and 5. First, in the Weyl representation, the covariant four-component equation for $\Psi_+ = [\,\varphi_1 \;\, 0\,]^{\mathrm{T}}$ (Eq. (53)) leads us to the following explicitly covariant two-component equation for the two-component wave function φ_1:

$$\hat{\eta}^{\mu}\partial_{\mu}\varphi_1 - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_2\varphi_1^* = 0, \tag{57}$$

where the matrices $\hat{\eta}^0 = -\mathrm{i}\hat{\sigma}_y$, $\hat{\eta}^1 = -\hat{\sigma}_z$, $\hat{\eta}^2 = -\mathrm{i}\hat{1}_2$, and $\hat{\eta}^3 = \hat{\sigma}_x$, satisfy the relation

$$(\hat{\eta}^{\mu})^*\,\hat{\eta}^{\nu} + (\hat{\eta}^{\nu})^*\,\hat{\eta}^{\mu} = -2g^{\mu\nu}\hat{1}_2 \tag{58}$$

(this last relation arises from Eq. (54)). After multiplying Eq. (57) by $-\hat{\sigma}_y$, this equation takes an alternative form, namely,

$$\mathrm{i}\hat{\sigma}^{\mu}\partial_{\mu}\varphi_1 + \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\sigma}_y\varphi_1^* = 0, \tag{59}$$

where $\hat{\sigma}^0 = \hat{1}_2$, $\hat{\sigma}^1 = \hat{\sigma}_x$, $\hat{\sigma}^2 = \hat{\sigma}_y$, and $\hat{\sigma}^3 = \hat{\sigma}_z$ (or, as it is commonly written, $\hat{\sigma}^{\mu} = (\hat{1}_2, +\hat{\boldsymbol{\sigma}})$) [8]. Equation (59) is precisely Eq. (28), as expected. Now, if we use the Majorana condition (Eq. (52)), we can obtain $\Psi_- = [\,0\ \ \varphi_2\,]^{\mathrm{T}}$ from $\Psi_+ = [\,\varphi_1\ \ 0\,]^{\mathrm{T}}$, and the result is $\Psi_- = [\,0\ \ \hat{\sigma}_y\,\varphi_1^*\,]^{\mathrm{T}}$ (which is in agreement with the result in Eq. (27)). Finally, we can write the four-component wave function for the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_- = [\,\varphi_1\ \ \hat{\sigma}_y\,\varphi_1^*\,]^{\mathrm{T}}$. It is clear that this four-component solution is dependent only on the two-component complex wave function φ_1, which is the solution of Eq. (59), i.e., here, we have only four independent real quantities. Because we have $\hat{\gamma}^5\Psi_+ = (+1)\Psi_+$, Eq. (59) is referred to as the right-chiral two-component Majorana equation.

Similarly, the covariant four-component equation for $\Psi_- = [\,0\ \ \varphi_2\,]^{\mathrm{T}}$ (Eq. (55)) leads us to the following explicitly covariant two-component equation for the two-component wave function φ_2:

$$\hat{\xi}^{\mu}\partial_{\mu}\varphi_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_2\varphi_2^* = 0, \tag{60}$$

where the matrices $\hat{\xi}^0 = -\hat{\eta}^0$, $\hat{\xi}^j = \hat{\eta}^j$, with $j = 1, 2, 3$, also satisfy Eq. (58) (in this case, the latter relation arises from Eq. (56)). Multiplying Eq. (60) by $\hat{\sigma}_y$, this equation takes the alternative form

$$\mathrm{i}\hat{\bar{\sigma}}^{\mu}\partial_{\mu}\varphi_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{\sigma}_y\varphi_2^* = 0, \tag{61}$$

where $\hat{\bar{\sigma}}^0 = \hat{\sigma}^0$, $\hat{\bar{\sigma}}^1 = -\hat{\sigma}^1$, $\hat{\bar{\sigma}}^2 = -\hat{\sigma}^2$, and $\hat{\bar{\sigma}}^3 = -\hat{\sigma}^3$ (i.e., $\hat{\bar{\sigma}}^{\mu} = (\hat{1}_2, -\hat{\boldsymbol{\sigma}})$). Equation (61) is precisely Eq. (29), as expected. Again, if we use the Majorana condition (Eq. (52)), we can obtain $\Psi_+ = [\,\varphi_1\ \ 0\,]^{\mathrm{T}}$ from $\Psi_- = [\,0\ \ \varphi_2\,]^{\mathrm{T}}$, and the result is $\Psi_+ = [\,-\hat{\sigma}_y\,\varphi_2^*\ \ 0\,]^{\mathrm{T}}$ (which is in agreement with the result in Eq. (27)). Thus, we can write the four-component wave function for the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_- = [\,-\hat{\sigma}_y\,\varphi_2^*\ \ \varphi_2\,]^{\mathrm{T}}$. The latter four-component solution depends only on the two-component complex wave function φ_2, which is the solution of Eq. (61), i.e., here, we have only four independent real quantities, as expected for a Majorana particle. Because we have $\hat{\gamma}^5\Psi_- = (-1)\Psi_-$, Eq. (61) is referred to as the left-chiral two-component Majorana equation.

In summary, Eq. (59) alone can be considered as a Majorana equation for the Majorana particle, even for a particular type of Majorana particle. Likewise, Eq. (61) alone can also be considered as a Majorana equation for the Majorana particle, even for a Majorana particle different from the previous one (for example, with a different mass). Thus, Eqs. (59) and (61), although similar, are non-equivalent two-component equations. Specifically, this is because φ_1 and φ_2 transform in two precise and different ways under a Lorentz boost, i.e., they transform according to two inequivalent representations of the Lorentz group [9]. Certainly, Eqs. (59) and (61) tend toward the pair of Weyl equations when $\mathrm{m}c^2 = V_\mathrm{S} = 0$ (Eq. (30)). Equations (59) and (61) comprise the so-called two-component theory of Majorana particles [8].

Again, in the Weyl representation that we have considered in our paper ($\hat{\gamma}^0 = \hat{\beta} = -\hat{\sigma}_x \otimes \hat{1}_2$, $\hat{\boldsymbol{\gamma}} = \hat{\beta}\hat{\boldsymbol{\alpha}} = +\mathrm{i}\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$, and $\hat{\gamma}^5 = +\hat{\sigma}_z \otimes \hat{1}_2$), we used $\hat{S}_C = -\hat{\gamma}^2 = -\mathrm{i}\hat{\sigma}_y \otimes \hat{\sigma}_y$, but this is only because we decided to derive this result from Eq. (8) (with $\hat{S}$ given by Eq. (10)). We could, for example, write $\hat{S}_C = -\mathrm{i}\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$. In the latter case, the equations for φ_1 and φ_2 are simply Eqs. (59) and (61) with the following replacement: $\hat{\sigma}_y \to +\mathrm{i}\hat{\sigma}_y$, namely,

$$\mathrm{i}\hat{\sigma}^\mu \partial_\mu \varphi_1 + \frac{1}{\hbar c}(V_\mathrm{S} + \mathrm{m}c^2)\mathrm{i}\hat{\sigma}_y \varphi_1^* = 0, \tag{62}$$

and

$$\mathrm{i}\hat{\bar{\sigma}}^\mu \partial_\mu \varphi_2 - \frac{1}{\hbar c}(V_\mathrm{S} + \mathrm{m}c^2)\mathrm{i}\hat{\sigma}_y \varphi_2^* = 0. \tag{63}$$

Equations (62) and (63) are essentially Eqs. (71) and (70) given in Ref. [9], respectively. In effect, as already mentioned before, in that remarkable reference the matrices $\hat{\boldsymbol{\alpha}}' = +\hat{\sigma}_z \otimes \hat{\boldsymbol{\sigma}}$, $\hat{\gamma}^{0\,\prime} = \hat{\beta}' = +\hat{\sigma}_x \otimes \hat{1}_2$, and $\hat{\boldsymbol{\gamma}}' = \hat{\beta}'\hat{\boldsymbol{\alpha}}' = -\mathrm{i}\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$, with $[\,\varphi_1' \; \varphi_2'\,]^\mathrm{T} \equiv \left[\,\tilde{\psi} \;\; \psi\,\right]^\mathrm{T}$, were considered as the Weyl representation. These matrices and those used by us are related through the relations $\hat{\boldsymbol{\alpha}}' = \hat{S}\,\hat{\boldsymbol{\alpha}}\,\hat{S}^\dagger$, $\hat{\beta}' = \hat{S}\,\hat{\beta}\,\hat{S}^\dagger$, $\hat{\boldsymbol{\gamma}}' = \hat{S}\,\hat{\boldsymbol{\gamma}}\,\hat{S}^\dagger$, and $[\,\varphi_1' \; \varphi_2'\,]^\mathrm{T} = \hat{S}\,[\,\varphi_1 \; \varphi_2\,]^\mathrm{T}$, where $\hat{S} = \hat{\sigma}_z \otimes \hat{1}_2 = \hat{S}^\dagger$. In addition, the charge-conjugation matrices, $\hat{S}_C' = -\mathrm{i}\hat{\gamma}^{2\,\prime} = -\hat{\sigma}_y \otimes \hat{\sigma}_y$ and $\hat{S}_C = -\mathrm{i}\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$, are related by means of Eq. (7). Thus, Eq. (62) for $\varphi_1 = \varphi_1' \equiv \tilde{\psi}$, with $V_\mathrm{S} = 0$, is Eq. (71) of Ref. [9], and Eq. (63) for $\varphi_2 = -\varphi_2' \equiv -\psi$, also with $V_\mathrm{S} = 0$, is Eq. (70) of the same reference. Also, in Ref. [9], the former equation was appropriately named the right-chiral two-component Majorana equation, and the latter was named the left-chiral two-component Majorana equation.

Unsurprisingly, Eqs. (62) and (63) for φ_1 and φ_2 can be written jointly in the form

$$\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu}\Psi - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_4\Psi_C = 0, \tag{64}$$

where $\Psi = [\,\varphi_1\ \varphi_2\,]^{\mathrm{T}}$ and $\Psi_C \equiv \hat{S}_C\,\Psi^*$ with $\hat{S}_C = -\mathrm{i}\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$. Specifically, Eq. (64) is the (four-component) Majorana equation with a scalar potential (see the discussion on this equation in the introduction). However, if this equation is considered to describe a Majorana particle with a four-component wave function, $\Psi = [\,\varphi_1\ \varphi_2\,]^{\mathrm{T}}$, it should be remembered that, due to the Majorana condition, $\Psi = \Psi_C$, φ_1 and φ_2 are not independent two-component wave functions. Therefore, in this case, it would be sufficient to solve just one of the two two-component Majorana equations, and then, with the relation between φ_1 and φ_2, we could reconstruct the entire wave function Ψ. However, if Eq. (64) is considered to describe a Majoranon [7, 19], then the two two-component Majorana equations must be solved, the solutions of which are simply the top and bottom components of the wave function Ψ in Eq. (64). The latter result is somewhat unexpected. Similarly, Eqs. (62) and (63) for $\tilde{\psi}$ and ψ in Ref. [9], respectively, can also be combined into Eq. (64). In this case, we write Eq. (64) with the following replacements: $\Psi \to \Psi' \equiv \Psi_M$, $\Psi_C \to \Psi'_C \equiv \Psi^{\mathrm{c}}_M$, and $\hat{\gamma}^{\mu} \to \hat{\gamma}^{\mu\,\prime} \equiv \hat{\gamma}^{\mu}$, where $\Psi' = \left[\,\tilde{\psi}\ \ \psi\,\right]^{\mathrm{T}}$ and $\Psi'_C \equiv \hat{S}'_C\,\Psi'^{*}$ with $\hat{S}'_C = -\mathrm{i}\hat{\gamma}^{2\,\prime} = -\hat{\sigma}_y \otimes \hat{\sigma}_y$. In the present case, Eq. (64), with $V_{\mathrm{S}} = 0$, is Eq. (123) of Ref. [9].

The same Weyl representation used in Ref. [9] was used in Ref. [20], but here, $\hat{S}'_C = +\mathrm{i}\hat{\gamma}^{2\,\prime} = +\hat{\sigma}_y \otimes \hat{\sigma}_y$ was chosen. Thus, in this case, the Majorana equations for φ'_1 and φ'_2 ($= +\mathrm{i}\hat{\sigma}_y\,\varphi_1^{*\,\prime}$) can be obtained from Eqs. (62) and (63) by making the following substitutions: $\varphi_1 \to \varphi'_1$, $\varphi_2 \to \varphi'_2$, and $\hat{\sigma}_y \to -\hat{\sigma}_y$. The equation for $\varphi'_1 \equiv \phi$, with $+\mathrm{i}\hat{\sigma}_y\,\phi^* \equiv \hat{\mathrm{S}}\phi$, and $V_{\mathrm{S}} = 0$, is Eq. (13) of Ref. [20], and the equation for $\varphi'_2 = +\mathrm{i}\hat{\sigma}_y\,\phi^* \equiv \hat{\mathrm{S}}\phi$, with $V_{\mathrm{S}} = 0$, is Eq. (14) of the same reference. Incidentally, by linearizing the standard relativistic energy–momentum relation, and without recourse to the Dirac equation, a good derivation of the two-component Majorana equation for $\varphi'_1 \equiv \phi$, with $V_{\mathrm{S}} = 0$, was obtained in Ref. [20].

The following matrices are also a very common choice when introducing the Weyl representation: $\hat{\gamma}^{0\,\prime} = \hat{\beta}' = +\hat{\sigma}_x \otimes \hat{1}_2$, $\hat{\boldsymbol{\gamma}}' = \hat{\beta}'\hat{\boldsymbol{\alpha}}' = +\mathrm{i}\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$, and $\hat{\gamma}^{5\,\prime} = -\hat{\sigma}_z \otimes \hat{1}_2$, with the respective wave function written in the form $[\,\varphi'_1\ \varphi'_2\,]^{\mathrm{T}}$. These matrices and those used by us are related as follows: $\hat{\gamma}^{\mu\,\prime} = \hat{S}\,\hat{\gamma}^{\mu}\hat{S}^{\dagger}$, etc, and $[\,\varphi'_1\ \varphi'_2\,]^{\mathrm{T}} = \hat{S}\,[\,\varphi_1\ \varphi_2\,]^{\mathrm{T}}$, where $\hat{S} = \hat{\sigma}_y \otimes \hat{1}_2 = \hat{S}^{\dagger}$.

In addition, by substituting the latter matrix, and $\hat{S}_C = -\mathrm{i}\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$ into Eq. (7), we obtain $\hat{S}'_C = -\hat{\sigma}_y \otimes \hat{\sigma}_y = +\mathrm{i}\hat{\gamma}^{2\,\prime}$ (which is a typical choice when considering this Weyl representation). Then, the equations for φ'_1 and φ'_2 are given by

$$\mathrm{i}\hat{\bar{\sigma}}^{\mu}\partial_{\mu}\varphi'_1 + \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\mathrm{i}\hat{\sigma}_y\varphi'^{*}_1 = 0, \tag{65}$$

and

$$\mathrm{i}\hat{\bar{\sigma}}^{\mu}\partial_{\mu}\varphi'_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\mathrm{i}\hat{\sigma}_y\varphi'^{*}_2 = 0. \tag{66}$$

Equation (65) for $\varphi'_1 \equiv \omega$, with $V_{\mathrm{S}} = 0$, is precisely Eq. (107) of Ref. [3], but this equation is the left-chiral two-component Majorana equation because, in this Weyl representation, one has $\hat{\gamma}^{5\,\prime} = -\hat{\sigma}_z \otimes \hat{1}_2$. We mention in passing that the reference by Pal is a great tutorial article that addresses in detail the key connections between the Dirac, Majorana, and Weyl fields in (3+1) dimensions. Likewise, Eq. (65) for $\varphi'_1 \equiv \nu$, with $V_{\mathrm{S}} = 0$, is just Eq. (4.93) of the renowned book by Mohapatra and Pal [21]. Note that in the latter reference, the following notation was used: $\hat{\sigma}_{\mu} = (\hat{1}_2, +\hat{\boldsymbol{\sigma}}) \Rightarrow \hat{\sigma}^{\mu} = (\hat{1}_2, -\hat{\boldsymbol{\sigma}})$, and $\hat{\bar{\sigma}}_{\mu} = (\hat{1}_2, -\hat{\boldsymbol{\sigma}}) \Rightarrow \hat{\bar{\sigma}}^{\mu} = (\hat{1}_2, +\hat{\boldsymbol{\sigma}})$, instead of the most common notation that we use in this article (see Eqs. (59) and (61)). Moreover, in this reference, the Majorana condition was written as $\Psi = \exp(-\mathrm{i}\delta)\Psi_C$ instead of as $\Psi = \Psi_C$, which is our choice. Likewise, Eq. (65) for $\varphi'_1 \equiv \chi$, with $V_{\mathrm{S}} = 0$, is Eq. (9) of Ref. [22]. In the latter great reference, a coupled system of two left-chiral Majorana equations was constructed and used to study neutrino oscillations for two Majorana neutrino flavor states.

Second, in the Dirac representation, the covariant four-component equation for Ψ_+ (Eq. (53)) leads us to the covariant two-component Eq. (57) with the following replacement: $\varphi_1 \to \varphi + \chi$. Likewise, the Majorana condition in Eq. (52) leads us to Eq. (27) with the latter replacement plus the following: $\varphi_2 \to -\varphi + \chi$, namely, $-\varphi + \chi = \hat{\sigma}_y(\varphi + \chi)^*$ (Eq. (19)). Remember that the four-component wave functions in the Dirac and Weyl representations are related through the relation $[\,\varphi_1\;\varphi_2\,]^{\mathrm{T}} = \hat{S}\,[\,\varphi\;\chi\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (11). Thus, from Eq. (53), one obtains the two-component wave function $\varphi + \chi$, from which one can construct Ψ_+, and using the Majorana condition, one obtains $-\varphi + \chi$, from which one can construct Ψ_- (see Table 3). Finally, the four-component wave function for the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_- = [\,\varphi\;\;\chi\,]^{\mathrm{T}}$, can be written immediately. Similarly, the covariant four-component equation for Ψ_- (Eq. (55)) leads us to Eq. (60) with the following replacement: $\varphi_2 \to -\varphi + \chi$. Thus, from Eq. (55), one obtains the two-component wave function $-\varphi + \chi$, from which one can construct Ψ_-, and using the Majorana condition one obtains $\varphi + \chi$, from which one can construct Ψ_+ (see Table 3). Finally, the four-component wave function for the Majorana particle can be written immediately.

Alternatively, adding and subtracting the former equations that result from Eqs. (53) and (55) and once again using (conveniently) the Majorana condition given in Eq. (19), one obtains an equation for the two-component wave function φ, namely,

$$\hat{\eta}^0 \partial_0 \varphi + \sum_{k=1}^{3} \hat{\eta}^k \partial_k \left(\hat{\sigma}_y \, \varphi^*\right) + \frac{1}{\hbar c}(V_{\rm S} + {\rm m}c^2)\hat{\sigma}_y \, \varphi = 0, \tag{67}$$

and another equation for the two-component wave function χ, namely,

$$\hat{\xi}^0 \partial_0 \chi + \sum_{k=1}^{3} \hat{\xi}^k \partial_k \left(\hat{\sigma}_y \, \chi^*\right) + \frac{1}{\hbar c}(V_{\rm S} + {\rm m}c^2)\hat{\sigma}_y \, \chi = 0. \tag{68}$$

Certainly, Eq. (67) leads to Eq. (20), and Eq. (68) leads to Eq. (21). Likewise, from the solution of Eq. (67) or Eq. (68), and properly using in each case the Majorana condition (Eq. (19)), one can obtain the respective four-component wave function $\Psi = [\, \varphi \;\; \chi \,]^{\rm T}$.

Third, in the Majorana representation, the covariant four-component equation for Ψ_+ (Eq. (53)) is precisely Eq. (50), and the covariant four-component equation for Ψ_- (Eq. (55)) is precisely Eq. (49); additionally, the latter equation is the complex conjugate of the former equation. This is shown by the following results. Remember that, in this representation, $\hat{S}_C = \hat{1}_4$; therefore, $\hat{\Gamma}^\mu = \hat{\gamma}^\mu_-$, $\hat{\Lambda}^\mu = \hat{\gamma}^\mu_+$, and, from the Majorana condition in Eq. (52), we have $\Psi_- = \Psi_+^*$ (and therefore, $\Psi = \Psi_+ + \Psi_- = \Psi^*$, as expected). In this representation, one also has $\hat{\gamma}^\mu = -(\hat{\gamma}^\mu)^*$ and $\hat{\gamma}^5 = -(\hat{\gamma}^5)^*$, and therefore, $\hat{\gamma}^\mu_- = -(\hat{\gamma}^\mu_+)^*$. Thus, in the Majorana representation, the equation for the Majorana particle is essentially Eq. (50), where $\hat{\gamma}^\mu_- = -(\hat{\gamma}^\mu_+)^*$ and $\Psi_- = \Psi_+^*$ (in fact, substituting the latter relations in the complex conjugate equation of Eq. (50), one obtains Eq. (49)). Specifically, Eq. (50) leads us to Eq. (57) with the following replacement: $\varphi_1 \to (\hat{1}_2 + \hat{\sigma}_y)\phi_1 - (\hat{1}_2 - \hat{\sigma}_y)\phi_2$. Remember that the four-component wave functions in the Majorana and Weyl representations are related by $[\, \varphi_1 \; \varphi_2 \,]^{\rm T} = \hat{S}^{-1} [\, \phi_1 \; \phi_2 \,]^{\rm T}$, where the matrix $\hat{S}$ is given in Eq. (10) (additionally, φ_1 and φ_2 are related by Eq. (27), i.e., the Majorana condition, which implies that ϕ_1 and ϕ_2 are real-valued wave functions, as expected). Finally, the equation obtained here and its complex conjugate can be written in the form given in Eq. (42).

In (1+1) dimensions. Let us introduce the following wave functions and matrices:

$$\Psi_{\pm} \equiv \frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right)\Psi\,, \quad \text{and} \quad \hat{\gamma}^{\mu}_{\pm} \equiv \frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right)\hat{\gamma}^{\mu}, \tag{69}$$

where the matrix $\hat{\Gamma}^5 \equiv -\mathrm{i}\hat{\gamma}^5$ is Hermitian because $\hat{\gamma}^5 \equiv \mathrm{i}\hat{\gamma}^0\hat{\gamma}^1 = \mathrm{i}\hat{\alpha}$ is anti-Hermitian, and satisfies the relations $(\hat{\Gamma}^5)^2 = \hat{1}_2$ and $\{\hat{\Gamma}^5, \hat{\gamma}^{\mu}\} = \hat{0}_2$. In addition, $\hat{\Gamma}^5$ satisfies the relation $\hat{S}_C\,(\hat{\Gamma}^5)^*(\hat{S}_C)^{-1} = \hat{\Gamma}^5$ (which is different from the analogous relation that satisfies $\hat{\gamma}^5$ in (3+1) dimensions), and

$$\left[\frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right)\right]^2 = \frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right)\,, \quad \text{and} \quad \frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right)\frac{1}{2}\left(\hat{1}_2 \mp \hat{\Gamma}^5\right) = \hat{0}_2. \tag{70}$$

Simply note that in (1+1) dimensions, $\hat{\Gamma}^5 = \hat{\alpha}$ acts similar to the standard fifth gamma matrix in (3+1) dimensions, i.e., as the chirality matrix [23, 24]. However, in this case, the charge conjugate of the wave functions in (69) verify $(\Psi_{\pm})_C = (\Psi_C)_{\pm}$. Thus, although it is verified that $\hat{\Gamma}^5\Psi_{\pm} = (\pm 1)\Psi_{\pm}$, we now have that $\hat{\Gamma}^5(\Psi_{\pm})_C = (\pm 1)(\Psi_{\pm})_C$, i.e., $\Psi_{\pm}$ and $(\Psi_{\pm})_C$ are eigenstates of $\hat{\Gamma}^5$ with eigenvalues ± 1. The matrices $\hat{\Gamma}^5$ and the wave functions $\Psi_{\pm}$ in each of the three representations that we use in this article are shown in Tables 2 and 6, respectively.

First, note that by multiplying the Dirac equation in Eq. (1) (but particularized to the case of (1+1) dimensions) by $\frac{1}{2}(\hat{1}_2 + \hat{\Gamma}^5)$ from the left, we obtain the equation

$$\mathrm{i}\hat{\gamma}^{\mu}_{+}\partial_{\mu}\Psi_{-} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_2\Psi_{+} = 0, \tag{71}$$

and similarly, multiplying Eq. (1) by $\frac{1}{2}(\hat{1}_2 - \hat{\Gamma}^5)$, we obtain the equation

$$\mathrm{i}\hat{\gamma}^{\mu}_{-}\partial_{\mu}\Psi_{+} - \frac{1}{\hbar c}(V_{\mathrm{S}} + \mathrm{m}c^2)\hat{1}_2\Psi_{-} = 0. \tag{72}$$

The latter pair of equations is completely equivalent to the Dirac equation and similar to the pair of Eqs. (49) and (50) in (3+1) dimensions. However, only in the present case, the gamma matrices in Eqs. (69) and (70) satisfy the relations

$$\hat{\gamma}^{\mu}_{\pm}\,\hat{\gamma}^{\nu}_{\mp} + \hat{\gamma}^{\nu}_{\pm}\,\hat{\gamma}^{\mu}_{\mp} = 2g^{\mu\nu}\frac{1}{2}\left(\hat{1}_2 \pm \hat{\Gamma}^5\right), \tag{73}$$

and $\{\hat{\gamma}^{\mu}_{+}, \hat{\gamma}^{\nu}_{+}\} = \{\hat{\gamma}^{\mu}_{-}, \hat{\gamma}^{\nu}_{-}\} = \hat{0}_2$. The charge-conjugate wave function also satisfies the Dirac equation; thus, we also have two equations equivalent to the latter equation. Specifically, by multiplying the Dirac equation for Ψ_C by $\frac{1}{2}(\hat{1}_2+\hat{\Gamma}^5)$ and $\frac{1}{2}(\hat{1}_2-\hat{\Gamma}^5)$, we obtain

$$\mathrm{i}\hat{\gamma}_+^\mu\partial_\mu(\Psi_-)_C-\frac{1}{\hbar c}(V_\mathrm{S}+\mathrm{m}c^2)\hat{1}_2(\Psi_+)_C=0\,,\quad \text{and}\quad \mathrm{i}\hat{\gamma}_-^\mu\partial_\mu(\Psi_+)_C-\frac{1}{\hbar c}(V_\mathrm{S}+\mathrm{m}c^2)\hat{1}_2(\Psi_-)_C=0, \tag{74}$$

respectively (remember that $(\Psi_\pm)_C=\hat{S}_C\Psi_\pm^*$). Note that just as Ψ_- and Ψ_+ satisfy Eqs. (71) and (72), $(\Psi_C)_-$ and $(\Psi_C)_+$ also satisfy them (this is because $(\Psi_\pm)_C=(\Psi_C)_\pm$). In the case of $\mathrm{m}c^2=V_\mathrm{S}=0$, we obtain $\mathrm{i}\hat{\gamma}_+^\mu\partial_\mu\Psi_-=\mathrm{i}\hat{\gamma}_+^\mu\partial_\mu(\Psi_-)_C=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu\Psi_-=\mathrm{i}\hat{\gamma}^\mu\partial_\mu(\Psi_-)_C=0$) and $\mathrm{i}\hat{\gamma}_-^\mu\partial_\mu\Psi_+=\mathrm{i}\hat{\gamma}_-^\mu\partial_\mu(\Psi_+)_C=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu\Psi_+=\mathrm{i}\hat{\gamma}^\mu\partial_\mu(\Psi_+)_C=0$).

The Majorana condition imposed upon the two-component wave function Ψ gives us the following relations:

$$\Psi_+=(\Psi_+)_C\quad \text{and}\quad \Psi_-=(\Psi_-)_C, \tag{75}$$

i.e., $\Psi_+=(\Psi_C)_+$ and $\Psi_-=(\Psi_C)_-$. Thus, unlike what happens in (3+1) dimensions, Ψ_+ and Ψ_- satisfy the Majorana condition. Clearly, the equation that describes a Majorana particle in (1+1) dimensions is the pair of Eqs. (71) and (72) (with the matrix relations (73)) and the pair of relations, or restrictions, in (75) (the Majorana condition). Naturally, by imposing the latter condition upon the equations in (74), we again obtain Eqs. (71) and (72).

On the other hand, making $\mathrm{m}c^2=V_\mathrm{S}=0$ in Eq. (71) leads us to the relation $\mathrm{i}\hat{\gamma}_+^\mu\partial_\mu\Psi_-=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu\Psi_-=0$), and as can be seen in Eq. (74), we also have $\mathrm{i}\hat{\gamma}_+^\mu\partial_\mu(\Psi_-)_C=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu(\Psi_-)_C=0$); in this case also, we have $\Psi_-=(\Psi_-)_C$ (this is due to the Majorana condition). Similarly, making $\mathrm{m}c^2=V_\mathrm{S}=0$ in Eq. (72) leads us to the relation $\mathrm{i}\hat{\gamma}_-^\mu\partial_\mu\Psi_+=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu\Psi_+=0$), but from Eq. (74) we also have $\mathrm{i}\hat{\gamma}_-^\mu\partial_\mu(\Psi_+)_C=0$ ($\Rightarrow \mathrm{i}\hat{\gamma}^\mu\partial_\mu(\Psi_+)_C=0$), where $\Psi_+=(\Psi_+)_C$ (because of the Majorana condition).

Thus, to obtain the two-component wave function that describes the one-dimensional Majorana particle, $\Psi=\Psi_++\Psi_-$, we must solve the system of equations formed by Eqs. (71) and (72), but Ψ_+ and Ψ_- must verify the relations in Eq. (75), i.e., the Majorana condition. Note that $\Psi=\Psi_++\Psi_-=(\Psi_+)_C+(\Psi_-)_C$, and therefore, $\Psi=\Psi_C$, as expected.

We can prove the following results. We make full use of Table 6. First, in the Weyl representation, the covariant Eq. (71) for the two-component wave functions $\Psi_+=[\,\varphi_1\;\;0\,]^\mathrm{T}$ and $\Psi_-=[\,0\;\;\varphi_2\,]^\mathrm{T}$ leads us only to an equation for the one-component wave functions φ_1 and φ_2, namely,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_2=+\mathrm{i}\hbar c\frac{\partial}{\partial x}\varphi_2+(V_\mathrm{S}+\mathrm{m}c^2)\varphi_1, \tag{76}$$

and similarly, the covariant equation (72) leads us to

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_1 = -\mathrm{i}\hbar c\frac{\partial}{\partial x}\varphi_1 + (V_\mathrm{S} + \mathrm{m}c^2)\varphi_2. \tag{77}$$

The latter pair of equations comprises a complex system of coupled equations; it is just Eq. (33), as expected. Likewise, the Majorana condition in Eq. (75) leads us to the pair of relations in Eq. (34), respectively, also as expected. Thus, we do not have a first-order equation for a single component of the wave function in the Weyl representation. Clearly, the four real degrees of freedom present in the solutions of Eqs. (76) and (77) are reduced to only two due to the two relations that emerge from the Majorana condition.

Incidentally, in (1+1) dimensions, one also has that φ_1 and φ_2 are transformed in two different ways under a Lorentz boost. In effect, let us write the Lorentz boost along the x-axis in the following way: $[\,ct'\; x'\,]^\mathrm{T} = \exp(-\omega\hat{\sigma}_x)\,[\,ct\; x\,]^\mathrm{T}$ (i.e., $x^{\mu\,\prime} = \Lambda^{\mu}{}_{\nu}\,x^{\nu}$), where, as usual, $\tanh(\omega) = v/c \equiv \beta$ and $\cosh(\omega) = (1-\beta^2)^{-1/2} \equiv \gamma$, with the speed of the primed (inertial) reference frame with respect to the unprimed (inertial) reference frame being v. Then, under this Lorentz boost, the wave function transforms as $\Psi'(x',t') = \hat{S}(\Lambda)\Psi(x,t)$, where $\hat{S}(\Lambda) = \exp(-\omega\hat{\Gamma}^5/2)$ and which obeys the relation $\Lambda^{\mu}{}_{\nu}\hat{\gamma}^{\nu} = \hat{S}^{-1}(\Lambda)\hat{\gamma}^{\mu}\hat{S}(\Lambda)$. Then, just in the Weyl representation, the matrix $\hat{S}(\Lambda)$ is a diagonal matrix, and we obtain the following results:

$$\varphi_1'(x',t') = \left[\cosh\left(\frac{\omega}{2}\right) - \sinh\left(\frac{\omega}{2}\right)\right]\varphi_1(x,t)\,,\;\; \varphi_2'(x',t') = \left[\cosh\left(\frac{\omega}{2}\right) + \sinh\left(\frac{\omega}{2}\right)\right]\varphi_2(x,t). \tag{78}$$

Thus, we have two different kinds of one-component wave functions in (1+1) dimensions. Certainly, not only do φ_1 and φ_2 satisfy the relations in (78) but also $(\varphi_1)_C$ and $(\varphi_2)_C$. This is because Ψ and Ψ_C are similarly transformed under the Lorentz boost (i.e., $\Psi_C'(x',t') = \hat{S}(\Lambda)\Psi_C(x,t)$). Interestingly, in the case where $\mathrm{m}c^2 = V_\mathrm{S} = 0$, the wave functions with definite chirality, Ψ_+ and Ψ_-, each satisfy the one-dimensional Dirac equation and their own Majorana conditions. Also, in the Weyl representation, the nonzero component of each of these two chiral wave functions satisfies a Weyl equation (see Eqs. (76) and (77)). Thus, we could call the particles described by Ψ_+ and Ψ_- Weyl-Majorana particles [25].

Second, in the Dirac representation, Eq. (71) leads us to Eq. (76) and Eq. (72) leads us to Eq. (77) with the following replacements: $\varphi_1 \to \varphi + \chi$ and $\varphi_2 \to \varphi - \chi$. Likewise, the Majorana condition (Eq. (75)) leads us precisely to the pair of relations in Eq. (34) with the latter replacements, namely, $\varphi + \chi = -\mathrm{i}(\varphi + \chi)^*$ and $\varphi - \chi = +\mathrm{i}(\varphi - \chi)^*$ (the latter two relations imply the result given in Eq.(23)). Remember that the two-component wave functions in the Dirac and Weyl representations are related through the relation $[\,\varphi_1\,\varphi_2\,]^{\mathrm{T}} = \hat{S}\,[\,\varphi\,\chi\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (15). Certainly, adding and subtracting the two equations obtained here and conveniently using the Majorana condition again, we obtain an equation for the component φ of the wave function, namely, Eq. (24), and an equation for the component χ of the wave function, namely, the same Eq. (24) but with the following replacements: $\varphi \to \chi$ and $V_{\mathrm{S}} + \mathrm{m}c^2 \to -(V_{\mathrm{S}} + \mathrm{m}c^2)$. As explained before, it is sufficient to solve only one of these two equations because the remaining respective component can be obtained from the Majorana condition. Thus, only two real quantities, or real degrees of freedom, are sufficient to fully describe the Majorana particle.

Third, in the Majorana representation, Eq. (71) leads us to Eq. (76) and Eq. (72) leads us to Eq. (77) with the following replacements: $\varphi_1 \to (1-\mathrm{i})(\phi_1+\phi_2)$ and $\varphi_2 \to (1+\mathrm{i})(\phi_1-\phi_2)$. Remember that the two-component wave functions in the Majorana and Weyl representations are related by $[\,\varphi_1\,\varphi_2\,]^{\mathrm{T}} = \hat{S}^{-1}\,[\,\phi_1\,\phi_2\,]^{\mathrm{T}}$, where the matrix $\hat{S}$ is given in Eq. (14). In this representation, $\hat{S}_C = \hat{1}_2$; therefore, the Majorana representation (Eq. (75)) is simply $\Psi_+ = \Psi_+^*$ and $\Psi_- = \Psi_-^*$, i.e., the latter condition inmediately yields the pair of relations $\phi_1+\phi_2 = \phi_1^*+\phi_2^*$ and $\phi_1-\phi_2 = \phi_1^*-\phi_2^*$, respectively (which implies the result in Eq. (43), i.e., the entire two-component wave function must be real). Finally, adding and subtracting the two equations obtained here (but before multiplying by $\mathrm{i}(1-\mathrm{i})$ the equation that arises from Eq. (76) and multiplying by $\mathrm{i}(1+\mathrm{i})$ the one that emerges from Eq. (77)), we obtain a real system of coupled equations, namely, the system in Eq. (44). Because the solutions of this system are real-valued, the wave function has two real degrees of freedom, as expected.

VI. CONCLUSIONS

Distinct differential equations can be used to describe a Majorana particle in (3+1) and (1+1) dimensions. We can have a complex single equation for a single component of the Dirac wave function, as it is in the Dirac and Weyl representations in (3+1) dimensions (in these cases, the single component itself is a two-component wave function), and in the Dirac representation in (1+1) dimensions (in this case, the single component itself is a one-component wave function). Apropos of this, in the Weyl representation in (3+1) dimensions, one can have two complex single equations, each being invariant under its own type of Lorentz transformation (or Lorentz boost), i.e., these two two-component covariant equations are non-equivalent equations, and each of them can describe a specific type of Majorana particle in (3+1) dimensions. Certainly, because of the Majorana condition, the solutions of these two equations are not independent of each other, that is, in the concrete description of the Majorana particle, two plus two (complex) components are not absolutely necessary (i.e., the solution of only one of the two two-component Majorana equations is what is needed to fully describe each type of Majorana particle). Unexpectedly, in the Weyl representation in (1+1) dimensions, we have a complex system of coupled equations, i.e., no first-order equation for any of the components of the wave function can be written. On the other hand, we can also have a real system of coupled equations, as it is in the Majorana representation in (3+1) and (1+1) dimensions.

All these equations and systems of equations emerge from the Dirac equation and the Majorana condition when a representation is chosen. Certainly, both the Dirac equation and the Majorana condition look different written in their component forms when different representations are used. In any case, whichever equation or system of equations is used to describe the Majorana particle, the wave function that describes it in (3+1) or (1+1) dimensions is determined by four or two real quantities (real components, real and imaginary parts of complex components, or just real or just imaginary parts of complex components), i.e., only four or two real quantities are sufficient.

Likewise, in (3+1) dimensions, the algebraic procedure introduced by Case (and reexamined by us) allows us to write two covariant equations of four components for the Majorana particle, i.e., in a form independent of the choice of a particular representation for the matrices $\hat{\Gamma}^{\mu}$ and $\hat{\Lambda}^{\mu}$ (see Eqs. (53) and (55)). Each of these equations provides one of the two covariant two-component Majorana equations that arise when choosing the Weyl representation. In contrast, in (1+1) dimensions, the algebraic procedure introduced by us leads only to a covariant system of coupled first-order equations of two components, and these components have their complex degrees of freedom restricted by two conditions that arise from the Majorana condition. This system of equations immediately gives us the complex system of coupled first-order equations of one component that emerges when using the Weyl representation, with the restriction given by the Majorana condition. However, in the Dirac representation, the same system of equations, together with the Majorana condition, can lead us to two one-component equations (each for a single component of the two-component wave function).

It is hoped that our results can be useful to enrich the subject of the distinct equations that can arise when describing the Majorana particle in (1+1) and (3+1) dimensions. As we have seen, the results obtained in these two space-time dimensions are not completely analogous. It is to be expected that these results also present important differences with results in (2+1) dimensions. However, in the latter case other difficulties can also arise. Definitely, these issues should be treated in another publication.

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Representation	$\hat{\boldsymbol{\alpha}}$	$\hat{\beta} \equiv \hat{\gamma}^0$	$\hat{\beta}\hat{\boldsymbol{\alpha}} \equiv \hat{\boldsymbol{\gamma}}$	$\hat{\gamma}^5 \equiv \mathrm{i}\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3$	$\hat{S}_C = \hat{S}^\dagger\hat{S}^*$
Dirac	$\hat{\sigma}_x \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_z \otimes \hat{1}_2$	$\mathrm{i}\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_x \otimes \hat{1}_2$	$-\mathrm{i}\hat{\sigma}_y \otimes \hat{\sigma}_y$
Weyl	$\hat{\sigma}_z \otimes \hat{\boldsymbol{\sigma}}$	$-\hat{\sigma}_x \otimes \hat{1}_2$	$\mathrm{i}\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_z \otimes \hat{1}_2$	$-\mathrm{i}\hat{\sigma}_y \otimes \hat{\sigma}_y$
Majorana	Table 1.1	$\hat{\sigma}_x \otimes \hat{\sigma}_y$	Table 1.2	$\hat{\sigma}_z \otimes \hat{\sigma}_y$	$\hat{1}_2 \otimes \hat{1}_2$

Table 1

$$
\boxed{\begin{gathered}
\hat{\alpha}_1 = -\hat{\sigma}_x \otimes \hat{\sigma}_x \\
\hat{\alpha}_2 = \hat{\sigma}_z \otimes \hat{1}_2 \\
\hat{\alpha}_3 = -\hat{\sigma}_x \otimes \hat{\sigma}_z
\end{gathered}}
$$

Table 1.1

$$
\boxed{\begin{gathered}
\hat{\gamma}^1 = \mathrm{i}\hat{1}_2 \otimes \hat{\sigma}_z \\
\hat{\gamma}^2 = -\mathrm{i}\hat{\sigma}_y \otimes \hat{\sigma}_y \\
\hat{\gamma}^3 = -\mathrm{i}\hat{1}_2 \otimes \hat{\sigma}_x
\end{gathered}}
$$

Table 1.2

Representation	$\hat{\alpha}$	$\hat{\beta} \equiv \hat{\gamma}^0$	$\hat{\beta}\hat{\alpha} \equiv \hat{\gamma}^1$	$\hat{\Gamma}^5 \equiv -\mathrm{i}\hat{\gamma}^5 = \hat{\gamma}^0\hat{\gamma}^1$	$\hat{S}_C = \hat{S}^\dagger\hat{S}^*$
Dirac	$\hat{\sigma}_x$	$\hat{\sigma}_z$	$\mathrm{i}\hat{\sigma}_y$	$\hat{\sigma}_x$	$-\mathrm{i}\hat{\sigma}_x$
Weyl	$\hat{\sigma}_z$	$\hat{\sigma}_x$	$-\mathrm{i}\hat{\sigma}_y$	$\hat{\sigma}_z$	$-\mathrm{i}\hat{\sigma}_z$
Majorana	$\hat{\sigma}_x$	$\hat{\sigma}_y$	$-\mathrm{i}\hat{\sigma}_z$	$\hat{\sigma}_x$	$\hat{1}_2$

Table 2

Representation	Ψ_+	Ψ_-
Dirac	$\frac{1}{2}\begin{bmatrix}\varphi+\chi\\ \varphi+\chi\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\varphi-\chi\\ -\varphi+\chi\end{bmatrix}$
Weyl	$\begin{bmatrix}\varphi_1\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ \varphi_2\end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix}(\hat{1}_2+\hat{\sigma}_y)\phi_1\\ (\hat{1}_2-\hat{\sigma}_y)\phi_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}(\hat{1}_2-\hat{\sigma}_y)\phi_1\\ (\hat{1}_2+\hat{\sigma}_y)\phi_2\end{bmatrix}$

Table 3

Representation	$\hat{\Gamma}^0$	$\hat{\Gamma}^1$	$\hat{\Gamma}^2$	$\hat{\Gamma}^3$
Dirac	$\frac{1}{2}\begin{bmatrix} -\hat{\sigma}_y & -\hat{\sigma}_y \\ -\hat{\sigma}_y & -\hat{\sigma}_y \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \mathrm{i}\hat{\sigma}_z & \mathrm{i}\hat{\sigma}_z \\ \mathrm{i}\hat{\sigma}_z & \mathrm{i}\hat{\sigma}_z \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\hat{1}_2 & -\hat{1}_2 \\ -\hat{1}_2 & -\hat{1}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\mathrm{i}\hat{\sigma}_x & -\mathrm{i}\hat{\sigma}_x \\ -\mathrm{i}\hat{\sigma}_x & -\mathrm{i}\hat{\sigma}_x \end{bmatrix}$
Weyl	$\begin{bmatrix} -\hat{\sigma}_y & \hat{0}_2 \\ \hat{0}_2 & \hat{0}_2 \end{bmatrix}$	$\begin{bmatrix} \mathrm{i}\hat{\sigma}_z & \hat{0}_2 \\ \hat{0}_2 & \hat{0}_2 \end{bmatrix}$	$\begin{bmatrix} -\hat{1}_2 & \hat{0}_2 \\ \hat{0}_2 & \hat{0}_2 \end{bmatrix}$	$\begin{bmatrix} -\mathrm{i}\hat{\sigma}_x & \hat{0}_2 \\ \hat{0}_2 & \hat{0}_2 \end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix} \hat{0}_2 & \hat{\sigma}_y - \hat{1}_2 \\ \hat{\sigma}_y + \hat{1}_2 & \hat{0}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \mathrm{i}\hat{\sigma}_z + \hat{\sigma}_x & \hat{0}_2 \\ \hat{0}_2 & \mathrm{i}\hat{\sigma}_z - \hat{\sigma}_x \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \hat{0}_2 & -\hat{\sigma}_y + \hat{1}_2 \\ \hat{\sigma}_y + \hat{1}_2 & \hat{0}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\mathrm{i}\hat{\sigma}_x + \hat{\sigma}_z & \hat{0}_2 \\ \hat{0}_2 & -\mathrm{i}\hat{\sigma}_x - \hat{\sigma}_z \end{bmatrix}$

Table 4

Representation	$\hat{\Lambda}^0$	$\hat{\Lambda}^1$	$\hat{\Lambda}^2$	$\hat{\Lambda}^3$
Dirac	$\frac{1}{2}\begin{bmatrix} \hat{\sigma}_y & -\hat{\sigma}_y \\ -\hat{\sigma}_y & \hat{\sigma}_y \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \mathrm{i}\hat{\sigma}_z & -\mathrm{i}\hat{\sigma}_z \\ -\mathrm{i}\hat{\sigma}_z & \mathrm{i}\hat{\sigma}_z \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\hat{1}_2 & \hat{1}_2 \\ \hat{1}_2 & -\hat{1}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\mathrm{i}\hat{\sigma}_x & \mathrm{i}\hat{\sigma}_x \\ \mathrm{i}\hat{\sigma}_x & -\mathrm{i}\hat{\sigma}_x \end{bmatrix}$
Weyl	$\begin{bmatrix} \hat{0}_2 & \hat{0}_2 \\ \hat{0}_2 & \hat{\sigma}_y \end{bmatrix}$	$\begin{bmatrix} \hat{0}_2 & \hat{0}_2 \\ \hat{0}_2 & \mathrm{i}\hat{\sigma}_z \end{bmatrix}$	$\begin{bmatrix} \hat{0}_2 & \hat{0}_2 \\ \hat{0}_2 & -\hat{1}_2 \end{bmatrix}$	$\begin{bmatrix} \hat{0}_2 & \hat{0}_2 \\ \hat{0}_2 & -\mathrm{i}\hat{\sigma}_x \end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix} \hat{0}_2 & \hat{\sigma}_y + \hat{1}_2 \\ \hat{\sigma}_y - \hat{1}_2 & \hat{0}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \mathrm{i}\hat{\sigma}_z - \hat{\sigma}_x & \hat{0}_2 \\ \hat{0}_2 & \mathrm{i}\hat{\sigma}_z + \hat{\sigma}_x \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \hat{0}_2 & -\hat{\sigma}_y - \hat{1}_2 \\ \hat{\sigma}_y - \hat{1}_2 & \hat{0}_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -\mathrm{i}\hat{\sigma}_x - \hat{\sigma}_z & \hat{0}_2 \\ \hat{0}_2 & -\mathrm{i}\hat{\sigma}_x + \hat{\sigma}_z \end{bmatrix}$

Table 5

Representation	Ψ_+	Ψ_-	$\hat{\gamma}^0_+ (= -\hat{\gamma}^1_+)$	$\hat{\gamma}^0_- (= \hat{\gamma}^1_-)$
Dirac	$\frac{1}{2}\begin{bmatrix} \varphi + \chi \\ \varphi + \chi \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \varphi - \chi \\ -\varphi + \chi \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
Weyl	$\begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \varphi_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix} \phi_1 + \phi_2 \\ \phi_1 + \phi_2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} \phi_1 - \phi_2 \\ -\phi_1 + \phi_2 \end{bmatrix}$	$\frac{\mathrm{i}}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$	$\frac{\mathrm{i}}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

Table 6